

Some notes on Problem 1.52 and determining vector potentials

Since I made enough sign errors to scare off most of you. I thought I would briefly outline my solution without all the sign errors. So the situation was as follows. We had two vector fields

$$\begin{aligned}\vec{v}_1 &= x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z} \\ \vec{v}_2 &= y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}\end{aligned}$$

We determined that the only vector field with no divergence was \mathbf{v}_1 . This means there should exist a vector potential \mathbf{W} such that

$$\vec{\nabla} \times \vec{W} = \vec{v}_1$$

Given the definition of curl, this means whichever solution for the vector potential \mathbf{W} we find (and you should be aware there are an **infinite** number of such solutions, as long as we add a vector field with no divergence to \mathbf{W} , the new vector field will also work as a vector potential solution):

$$\begin{aligned}\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} &= v_{1x} = x^2 \\ \frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} &= v_{1y} = 3xz^2 \\ \frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} &= v_{1z} = -2xz\end{aligned}$$

So at this stage, by first approach was brute force. Try to take each line in the above equations and assume “half” of the right hand side is due to each of the left hand side derivatives... such that:

$$\begin{aligned}\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} &= x^2 &\rightarrow W_z &= \frac{1}{2}x^2y + f(x, z), & W_y &= -\frac{1}{2}x^2z + g(x, y) \\ \frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} &= 3xz^2 &\rightarrow W_x &= \frac{1}{2}xz^3 + h(x, y), & W_z &= -\frac{3}{4}x^2z^2 + j(y, z) \\ \frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} &= -2xz &\rightarrow W_y &= -\frac{1}{2}x^2z + k(y, z), & W_x &= xyz + m(x, z)\end{aligned}$$

Notice we have a problem here in the W_x term, since we have a term with xyz in it... this is troublesome because $h(x,y)$ suggests no such term should exist. Same with W_z terms. Notice the two expressions for W_x are actually consistent! So at this stage, experience has shown be that eliminating one of the troublesome terms might be useful. We can choose from either W_x or W_z . I'll do both here to show you the results.

APPROACH #1: Since W_x is a pain, set it to zero and let's re-evaluate:

$$\begin{aligned} \frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} &= x^2 &\rightarrow W_z &= \frac{1}{2}x^2y + f(x, z), & W_y &= -\frac{1}{2}x^2z + g(x, y) \\ \frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} &= 3xz^2 &\rightarrow W_x &= 0, & W_z &= -\frac{3}{2}x^2z^2 + j(y, z) \\ \frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} &= -2xz &\rightarrow W_y &= -x^2z + k(y, z), & W_x &= 0 \end{aligned}$$

Notice this doesn't work because there W_z components are now inconsistent and W_y have inconsistent coefficients. However, I notice if I take $W_y = -x^2z$, then I could assume $dW_z/dy = 0$, and assume no y related component to W_z and then everything works with:

$$\vec{W} = -x^2z\hat{y} - \frac{3}{2}x^2z^2\hat{z}$$

Just for sanity's sake, I confirm the curl of this vector potential is indeed \mathbf{v}_1 :

$$\begin{aligned} \vec{\nabla} \times \vec{W} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x^2z & -\frac{3}{2}x^2z^2 \end{vmatrix} \\ &= x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z} \end{aligned}$$

APPROACH #2: Since W_z is a pain, set it to zero and let's re-evaluate:

$$\begin{aligned} \frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z} &= x^2 &\rightarrow W_z &= 0, & W_y &= -x^2z + g(x, y) \\ \frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x} &= 3xz^2 &\rightarrow W_x &= xz^3 + h(x, y), & W_z &= 0 \\ \frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y} &= -2xz &\rightarrow W_y &= -\frac{1}{2}x^2z + k(y, z), & W_x &= xyz + l(x, z) \end{aligned}$$

Notice the W_x components are now inconsistent and W_y have inconsistent coefficients. But as I did with approach #1, I notice if I take $W_y = -x^2z$, then I could assume $dW_x/dy = 0$ and and assume no x related component to W_z and then everything works with:

$$\vec{W} = xz^3\hat{x} - x^2z\hat{y}$$

Notice this solution differs from our solution in approach #1, but its curl is the same:

$$\vec{\nabla} \times \vec{W} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -x^2z & 0 \end{vmatrix} = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$$