SPECIAL NOTE: I could not find a symbol that exactly matches the book's script r for the separation vector. Instead I am using the following notation: $\vec{\mathfrak{r}} = \vec{r} - \vec{r}'$, $\mathfrak{r} = |\vec{\mathfrak{r}}|$ and $\hat{\mathfrak{r}} = \vec{\mathfrak{r}}/\mathfrak{r}$.

1. **Griffiths Problem 3.28:** In Example 3.9 we derived the exact potential for a spherical shell of radius R, which carries a surface charge $\sigma = k\cos\theta$.
   (a) Calculate the dipole moment of this charge distribution.
   (b) Find the approximate potential, at points far from the sphere, and compare the exact answer $[V(r,\theta) = \frac{kR^2}{3\epsilon_0}\frac{1}{r^2}\cos\theta\ (r \geq R)]$. What can you conclude about the higher multipoles?

In this problem we consider a spherical shell of radius R which carries a surface charge distribution $\sigma = k\cos\theta$. We begin by calculating the dipole moment because the total charge on the sphere is zero. The dipole moment is given by

$$\vec{p} = \int \vec{r}'\rho d\tau'. \tag{1}$$

The x and y components of the dipole moment are zero because of the symmetry of the charge distribution. The z-component of the dipole moment is given by

$$p_z = \vec{p}\cdot\hat{z} = \int \vec{r}'\cdot\hat{z}\rho d\tau' = R\int \cos\theta'\rho d\tau' \tag{2}$$

Given this expression and the fact that $\rho d\tau' = dq' = \sigma da' = \sigma R^2 \sin\theta' d\theta' d\phi'$ I can now manipulate this

$$p_z = R\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} \sigma\cos\theta' R^2 \sin\theta' d\phi' d\theta' \tag{3a}$$

$$= 2\pi R^3 \int_{\theta='0}^{\pi} (k\cos\theta')\cos\theta'\sin\theta' d\theta' \tag{3b}$$

$$= 2\pi k R^3 \int_{\theta='0}^{\pi} \cos^2\theta'\sin\theta' d\theta' \tag{3c}$$

$$= -2\pi k R^3 \left.\frac{\cos^3\theta'}{3}\right|_0^{\pi} \tag{3d}$$

$$= \frac{4}{3}\pi k R^3; \tag{3e}$$

Once we have the z-component of the dipole, we can compute the dipole term in the potential of the sphere to be

$$V = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{kR^3\cos\theta}{3\epsilon_0 r^2}. \tag{4}$$

From Equation 3.87 in the textbook we know this is the exact potential for the sphere, so all the higher multipole moments must be zero. *(Thanks to Dr. Craig for providing the initial draft of this solution in LaTeX form.)*

2. **Griffiths Problem 3.33:** Show that the electric field of a ("pure") dipole [$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$, Eq. 3.103] can be written as
$$\vec{E}_{dip}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p})$$
Also clarify why this expression is referred to as the coordinate-free form of the dipole electric field. **HINT:** It might be useful to work backward from the expression above back to equation 3.103. It is also helpful to note that $\vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$.

The electric field of a "pure" dipole which points along the z-axis is given by equation 3.103 from the textbook:

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right). \tag{5}$$

We are to rewrite this in a coordinate-free form. For a dipole along the z-axis, $\vec{p}\cdot\hat{r} = p\cos\theta$ and $\vec{p}\cdot\hat{\theta} = -p\sin\theta$, With this the electric field becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(2(\vec{p}\cdot\hat{r})\hat{r} - (\vec{p}\cdot\hat{\theta})\hat{\theta}\right). \tag{6}$$

We can also write $\vec{p} = (\vec{p}\cdot\hat{r})\hat{r} + (\vec{p}\cdot\hat{\theta})\hat{\theta}$ so that $(\vec{p}\cdot\hat{\theta})\hat{\theta} = \vec{p} - (\vec{p}\cdot\hat{r})\hat{r}$. Putting this into the expression for the electric field (6) gives

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right) \tag{7}$$

This is referred to as the coordinate-free form of the dipole electric field for the reason that neither r nor θ appear explicitly in the equation, instead all that information is rolled up into the dipole moment vector $\vec{p}$. *(Thanks to Dr. Craig for providing the initial draft of this solution in LaTeX form.)*

3. **Griffiths Problem 3.40:** A thin insulating rod, running from $z = -a$ to $z = +a$, carries the indicated line charges. In each case, find the leading term in the multipole expansion of the potential

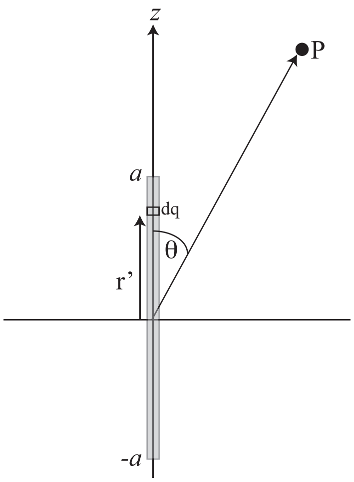
   (a) $\lambda = k\cos\left(\frac{\pi z}{2a}\right)$, where k is a constant.

   (b) $\lambda = k\sin\left(\frac{\pi z}{a}\right)$, where k is a constant.

   (c) $\lambda = k\cos\left(\frac{\pi z}{a}\right)$, where k is a constant.

   HINT: To know what is the leading term in the multipole expansion, it will be helpful to compute the total charge Q and if that is zero, to compute the dipole moment $\vec{p}$ and if that is zero, compute the quadropole moment. Also, consider what these linear charge densities λ actually mean in terms of where the charge is.

Consider the line of charge shown below.



In all three of these problems, we are asked to find the potential due to this line of charge at the point P using the multipole expansion. The small box in the figure highlights a charge element $dq = \rho d\tau' = \lambda dz'$

a distance r' from the origin. The multipole expansion for this charge element is

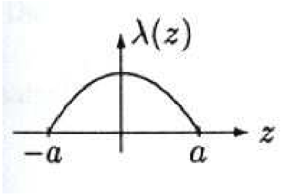
$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r}\int \rho d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3\cos^2\theta'}{2} - \frac{1}{2}\right)\rho d\tau' + \cdots\right). \tag{8}$$

The expansion can also be written in the form

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{1}{r^3}\int (r')^2\left(\frac{3\cos^2\theta'}{2} - \frac{1}{2}\right)\rho d\tau' + \cdots\right), \tag{9}$$

where Q is the total charge of the distribution and $\vec{p}$ is its dipole moment. For each of the three charge distributions given we are supposed to find the first nonzero term in the multipole expansion. As hinted, the strategy for each distribution will be to first calculate Q, and if it is zero to then calculate $\vec{p}$ and if that is zero then calculate the quadrupole term.

(a) The first charge distribution is $\lambda_1 = k\cos(\pi z/2a)$ and looks like



which suggests there should be a non-zero total charge. The total charge of this distribution is

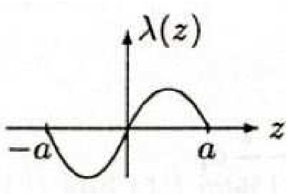
$$\begin{aligned}
Q &= \int_{-a}^{a} \lambda dz' && \text{(10a)}\\
&= k\int_{-a}^{a} \cos\left(\frac{\pi z'}{2a}\right) dz' && \text{(10b)}\\
&= 2k\int_{0}^{a} \cos\left(\frac{\pi z'}{2a}\right) dz' && \text{(10c)}\\
&= 2k\left[\left(\frac{2a}{\pi}\right)\sin\left(\frac{\pi z'}{2a}\right)\right]_0^a && \text{(10d)}\\
&= \frac{4ka}{\pi}. && \text{(10e)}
\end{aligned}$$

The approximate potential of the distribution is

$$V = \frac{Q}{4\pi\epsilon_0 r} = \frac{ka}{\pi^2\epsilon_0 r}. \tag{11}$$

In this case, since there was an overall charge, the first-order monopole moment will be adequate.

(b) The next charge distribution is $\lambda = k\sin(\pi z/a)$ and looks like



which suggests zero total charge and a non-zero dipole moment. Computing the total charge I find

$$Q = k\int_{-a}^{a} \sin\left(\frac{\pi z'}{a}\right) dz' = 0 \tag{12}$$

because the integrand is an odd function of z' and we are integrating over a symmetric interval. The dipole moment is

$$\vec{p} = \int \vec{r}' \rho d\tau'; \tag{13}$$

here the x and y components are zero by symmetry so the only (potentially) non-zero component is

$$p_z = \int z'\lambda dz'. \tag{14}$$

For the charge distribution in this part of the problem this is

$$p_z = \int_{-a}^{a} kz' \sin\left(\frac{\pi z'}{a}\right) dz' \tag{15a}$$

$$= 2k\int_{0}^{a} z' \sin\left(\frac{\pi z'}{a}\right) dz' \tag{15b}$$

$$= 2k\left(\frac{a}{\pi}\right)^2 \left[\sin\left(\frac{\pi z'}{a}\right) - \left(\frac{\pi z'}{a}\right)\cos\left(\frac{\pi z'}{a}\right)\right]_0^a \tag{15c}$$

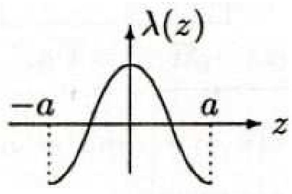
$$= \frac{2ka^2}{\pi}. \tag{15d}$$

The potential from the multipole expansion (9) is

$$V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{ka^2\cos\theta}{2\pi^2\epsilon_0 r^2}. \tag{16}$$

Here the monopole term is zero, and we don't calculate the quadrupole because the dipole term is not zero.

(c) The final charge distribution is $\lambda = k\cos(\pi z/a)$ and looks like



which suggests zero total charge and a zero dipole moment. I can verify its total charge is zero,

$$Q = k\int_{-a}^{a}\cos(\pi z'/a)dz' = \frac{ka}{\pi}\sin\left(\frac{\pi z'}{a}\right)\bigg|_{-a}^{a} = 0, \tag{17}$$

and that its dipole moment is also zero,

$$p_z = k\int_{-a}^{a} z'\cos\left(\frac{\pi z'}{a}\right)dz' = 0 \tag{18}$$

because the integrand is an odd function of z'. That means we have to evaluate the quadrupole moment

$$V \approx \frac{k}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3\cos^2\theta'}{2} - \frac{1}{2}\right)\rho d\tau' \tag{19a}$$

$$= \frac{k}{4\pi\epsilon_0 r^3}\frac{(3\cos\theta^2 - 1)}{2}\int_{-a}^{a}(z')^2\cos\left(\frac{\pi z'}{a}\right)dz'. \tag{19b}$$

The integral in (19b) is

$$\int_{-a}^{a}(z')^2\cos\left(\frac{\pi z'}{a}\right)dz' = 2\int_{0}^{a}(z')^2\cos\left(\frac{\pi z'}{a}\right)dz' \tag{20a}$$

$$= 2\left(\frac{a}{\pi}\right)^3\left[2\left(\frac{\pi z}{a}\right)\cos\left(\frac{\pi z'}{a}\right) + \left(\left(\frac{\pi z'}{a}\right)^2 - 2\right)\sin\left(\frac{\pi z'}{a}\right)\right]_0^a \tag{20b}$$

$$= -4\frac{a^3}{\pi^2}. \tag{20c}$$

With this the potential of the distribution becomes

$$V = \frac{ka^3}{2\pi^3\epsilon_0}\frac{3\cos^2\theta - 1}{r^3}. \tag{21}$$

(Thanks to Dr. Craig for providing the initial draft of this solution in LaTeX form.)

4. **Griffiths Problem 4.03:** According to equation 4.1 ($\vec{p} = \alpha\vec{E}$), the induced dipole moment of an atom, $\vec{p}$, is proportional to the external field, $\vec{E}$. This is a "rule of thumb," not a fundamental law, and it is easy to concoct exceptions — in theory. Suppose, for example, the charge density of the electron cloud were proportional to the distance from the center, out to radius R (*e.g.*- $\rho(r) = Ar$ where A is a constant with the right units). To what power of E would p be proportional to in that case? Find the condition on $\rho(r)$ such that Equation 4.1 will hold in the weak-field limit. **HINT:** Assume the electron cloud is spherically symmetric and determine a function for the electric field given the charge distribution mentioned. This "internal" electric field has to balance the external one when the nucleus is off center by some amount d. Understand Example 4.1 for guidance.

Start off by assuming $\rho(r) = Ar$ in spherical coordinates. Since this is a spherically symmetric charge distribution, we expect a spherically symmetric electric field. As such we can use a spherical Gaussian surface centered at the origin and apply the integral form of Gauss' law (equation 2.44) to determine the "internal" electric field due to this electron cloud:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \tag{22a}$$

$$E(4\pi r^2) = \frac{\int\limits_{r'=0}^{r} \int\limits_{\theta'=0}^{\pi} \int\limits_{\phi'=0}^{2\pi} Ar' r'^2 \sin\theta d\theta' d\phi' dr'}{\epsilon_0} \tag{22b}$$

$$E(4\pi r^2) = \frac{4\pi A}{\epsilon_0} \int\limits_{r'=0}^{r} r'^3 dr' \tag{22c}$$

$$E(4\pi r^2) = \frac{\pi A}{\epsilon_0} r^4 \tag{22d}$$

$$E_{int} = \frac{A}{4\epsilon_0} r^2 \tag{22e}$$

This "internal" electric field balances the "external" electric field when the nucleus is off-center by distance d such that:

$$E_{ext} = E_{int} = \frac{A}{4\epsilon_0} d^2. \tag{23}$$

Now let's solve for this distance d:

$$d^2 = \frac{4\epsilon_0 E_{ext}}{A} \tag{24a}$$

$$d = \sqrt{\frac{4\epsilon_0 E_{ext}}{A}} \tag{24b}$$

$$= 2\sqrt{\frac{\epsilon_0 E_{ext}}{A}} \tag{24c}$$

Given a charge of $q = e$, then the induced dipole moment is (based on equation 3.101) just $p = qd = ed = 2e\sqrt{\frac{\epsilon_0 E_{ext}}{A}}$. So $p \propto E_{ext}^{1/2}$, which is, of course, different than equation 4.1.

Now let's consider that "weak-field" case they want us to consider. Since $p = qd$, for equation 4.1 to hold, we need $p \propto E$ and therefore in the weak-field limit $E \propto d \propto r$ for small r (remember, "weak" field). Examination of Guass' Law above shows us that for $E \propto r$, we need $Q_{enc} \propto r^3$ which implies a constant charge density ρ. So the condition on $\rho(r)$ for equation 4.1 to hold in the "weak-field" case is $\rho(r) = \mathit{constant}$ for small values of r.

5. **Griffiths Problem 4.10 tweaked:** A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$, where k is a constant and $\vec{r}$ is the vector from the center.

   (a) Calculate the bound charges σ_b and ρ_b.

   (b) What is the meaning behind these "bound charges"? Are they actual charges and if not, why do we bother to compute them? In other words, of what use are they?

   (c) Find the electric field inside and outside the sphere. **HINT:** Answering part (b) first might help you with your approach this problem. Also, the solution to Griffiths 2.12 which was on Problem Set 3, might help.

(a) The bound charges for a given polarization $\vec{P}$ are given by the equations 4.11, $\sigma_b \equiv \vec{P}\cdot\hat{n}$, and 4.12, $\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$. Its a simple matter to note that for a sphere $\hat{n} = \hat{r}$ and so the bound surface charge density is

$$\sigma_b = \vec{P}\cdot\hat{n} \tag{25a}$$

$$= \left[\vec{P}\cdot\hat{r}\right]_{r=R} = \left[k\vec{r}\cdot\hat{r}\right]_{r=R} = \left[kr\hat{r}\cdot\hat{r}\right]_{r=R} \tag{25b}$$

$$= kR \tag{25c}$$

and the bound volume charge density

$$\rho_b = -\vec{\nabla}\cdot\vec{P} \tag{26a}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right) \quad \text{[using divergence of radial component in spherical coordinates]} \tag{26b}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^3\right) = -\frac{1}{r^2}3kr^2 \tag{26c}$$

$$= -3k. \tag{26d}$$

(b) The meaning of these *bound charges* is that the potential produced by an object of polarizaion $\vec{P}$ is the same as that produced by an object with volume charge density $\rho_b = -3k$ and surface charge density $\sigma_b = kR$. Its easier to compute with the electric field of a charge distribution than that of a bunch of infinitesimal dipoles that contribute to the polarization.

(c) For the electric field inside the sphere($r \leq R$) with polarization $\vec{P}(\vec{r}) = k\vec{r}$ should be equal to that of a sphere with a volume charge density equal to the bound volume charge density we computed in part (a). Using Guass' law in problem 2.12 we found that inside the sphere,

$$\vec{E} = \frac{\rho}{3\epsilon_0} r\hat{r}. \tag{27}$$

Using this solution, it is clear the electric field within the sphere is:

$$\vec{E} = \frac{-3k}{3\epsilon_0} r\hat{r} = \frac{-k}{\epsilon_0} r\hat{r}. \tag{28}$$

For outside the sphere ($r > R$), symmetry states the electric field must be radial, therefore we can use a spherical Gaussian surface to argue the electric field should be equal to to that of a sphere with the equivalent total charge, such that

$$\begin{aligned}
\oint \vec{E} \cdot d\vec{a} &= \frac{Q_{enc}}{\epsilon_0} && \text{(29a)}\\
E(4\pi r^2) &= \frac{\rho_b \frac{4}{3}\pi R^3 + \sigma_b 4\pi R^2}{\epsilon_0} && \text{(29b)}\\
E &= \frac{\frac{\rho_b}{3} R^3 + \sigma_b R^2}{\epsilon_0 r^2} = \frac{\frac{-3k}{3} R^3 + (kR) R^2}{\epsilon_0 r^2} && \text{(29c)}\\
&= 0 && \text{(29d)}
\end{aligned}$$

So the electric field outside this polarized sphere is zero!

6. **Griffiths Problem 4.14 tweaked:** When you polarize a neutral dielectric, charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges σ_b and ρ_b. Given the expressions for the bound charges from Equations 4.11 ($\sigma_b \equiv \vec{P} \cdot \hat{n}$) and 4.12 ($\rho_b \equiv -\vec{\nabla} \cdot \vec{P}$), develop an expression for the total charge and show that the total charge vanishes. What does this mean? **HINT:** If you want to do this painlessly, you'll want to exploit the divergence theorem.

The total charge should be the integral over area of the surface charge density, σ_b, plus the integral over volume of the volume charge density, ρ_b. Using this idea and equations 4.11 and 4.12, we can write the expression for total charge as:

$$Q_{tot} = \oint\limits_{\mathcal{S}} \sigma_b da + \int\limits_{\mathcal{V}} \rho_b d\tau \tag{30a}$$

$$= \oint\limits_{\mathcal{S}} \vec{P} \cdot \hat{n} da - \int\limits_{\mathcal{V}} \vec{\nabla} \cdot \vec{P} d\tau \tag{30b}$$

$$= \oint\limits_{\mathcal{S}} \vec{P} \cdot d\vec{a} - \int\limits_{\mathcal{V}} \vec{\nabla} \cdot \vec{P} d\tau \quad (\text{using } d\vec{a} = da\hat{n}) \tag{30c}$$

However, the divergence theorem applied to $\vec{\nabla} \cdot P$ states (equation 1.56) that:

$$\int\limits_{\mathcal{V}} (\vec{\nabla} \cdot P) d\tau = \oint\limits_{\mathcal{S}} \vec{P} \cdot d\vec{a} \tag{31}$$

therefore by looking at equation 30c we see the total charge must be zero!