**SPECIAL NOTE:** I could not find a symbol that exactly matches the book's script r for the separation vector. Instead I am using the following notation: $\vec{\mathfrak{r}} = \vec{r} - \vec{r}'$, $\mathfrak{r} = |\vec{\mathfrak{r}}|$ and $\hat{\mathfrak{r}} = \vec{\mathfrak{r}}/\mathfrak{r}$.

1. **Griffiths Problem 3.09 (variant):** A uniform line charge $\lambda$ is places on an infinite straight wire, a distance $d$ above a grounded conducting plane. (Lets say the wire runs parallel to the $x$-axis and directly above it, and the conducting plane is the $xy$ plane.)

   (a) Examine Problem 2.47 in the textbook, which has solution $V(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}$. Explain clearly why solving problem 2.47 is equivalent to solving the problem for the potential in the region above the plane. In the process, also explain clearly why it is *NOT* equivalent to solving for the potential below the plane!

   (b) Actually find the potential in the region above the plane! **HINT:** You did solve for the electric potential of a single infinite wire in Problem 2.22 which was on Problem Set 4. Use this information to quickly solve this problem.

   (c) Find the charge density $\sigma$ induced on the conducting plane.

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(a) In order for Problem 2.47 to have the same solution as this problem (above the plane), the boundary conditions have to match. Boundary Condition (1) is that since we are told the conducting plane is grounded, that means $V(x,y,0) = 0$. Boundary condition (2) comes from the fact that for the electric potential of a single wire as given in Problem 2.22 ($V(s) = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{s_0}{s}$) shows that $V(s \to \infty) \to 0$. Examining the solution to Problem 2.47,

$$V(x,y,z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\} \tag{1}$$

we see that at $z = 0$

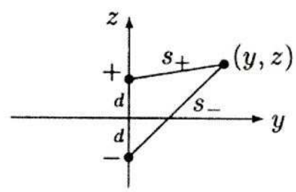
$$V(x,y,0) = \frac{\lambda}{4\pi\epsilon_0}\ln\left\{\frac{y^2+d^2}{y^2+d^2}\right\} = 0 \tag{2}$$

so it fulfills boundary condition (1). Also, if $y^2 + z^2 \gg d^2$ then equation 1 takes the form:

$$V(x, y, z) = \frac{\lambda}{4\pi\epsilon_0} \ln \left\{ \frac{y^2 + z^2 + 2zd + d^2}{y^2 + z^2 - 2zd + d^2} \right\} \rightarrow \frac{\lambda}{4\pi\epsilon_0} \ln \left\{ \frac{y^2 + z^2}{y^2 + z^2} \right\} \rightarrow 0. \tag{3}$$

So all the boundary conditions are met. By Uniqueness Theorem 1, this means that the potential in the volume above the grounded conducting plane is correctly described by the solution to the two wire potential. The reason we can't solve for the potential below the grounded conducting plane by this method is quite simple... we introduced an image charge in this region, which has changed its potential from it's original value.

(b) From the solution to Problem 2.22, we know the potential of a single infinitely long straight wire with linear charge density $\lambda$ is cylindrically symmetric and can be expressed $V(s) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{s_0}{s}$ where $s_0$ was the reference location. If I now consider the two oppositely-charged wires to be parallel to the $x$-axis, then I can sketch the situation as shown below:




Where the potential of the two wires together is:

$$V(y, z) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{s_0}{s_+} - \frac{\lambda}{2\pi\epsilon_0} \ln \frac{s_0}{s_-} \tag{4a}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \left\{ \frac{s_-}{s_+} \right\}. \tag{4b}$$

Now $s_+$ is the distance from the positively charged wire at position $y = 0, z = d$ to your location, which can be written $s_+ = \sqrt{y^2 + (z - d)^2}$. The $s_-$ term is the same for the negatively charged wire, therefore:

$$V(y,z) = \frac{\lambda}{2\pi\epsilon_0} \ln\left\{\frac{\sqrt{y^2+(z+d)^2}}{\sqrt{y^2+(z-d)^2}}\right\} \tag{5a}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\} \tag{5b}$$

(c) By equation 2.49 we know that for a surface charge in this case can be written

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} \tag{6}$$

so applying equation 6 to the potential in equation 5b we find

$$\sigma = -\epsilon_0 \frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}\right]_{z=0} \tag{7a}$$

$$= -\frac{\lambda}{4\pi}\frac{\partial}{\partial z}\left[\ln\left\{\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right\}\right]_{z=0} \tag{7b}$$

$$= -\frac{\lambda}{4\pi}\left[\frac{1}{y^2+(z+d)^2}2(z+d) - \frac{1}{y^2+(z-d)^2}2(z-d)\right]_{z=0} \tag{7c}$$

$$= -\frac{\lambda}{4\pi}\left[\frac{2d}{y^2+d^2} - \frac{-2d}{y^2+d^2}\right] \tag{7d}$$

$$\sigma(y) = -\frac{\lambda}{\pi}\frac{d}{y^2+d^2} \tag{7e}$$

We can confirm this by considering a segement of the wire of length $\ell$ such that the total charge on that segment is $q = \ell\lambda$. Recalling the writes are parallel to the $x$ axis, this means I can consider the total induced charge on a strip of the conducting surface parallel to the $y$-axis with length $\ell$ to get the corresponding induced charge:

$$q = \ell \int_{y=-\infty}^{\infty} -\frac{\lambda}{\pi} \frac{d}{y^2 + d^2} dy \tag{8a}$$

$$= -\frac{\ell \lambda d}{\pi} \int_{y=-\infty}^{\infty} \frac{dy}{y^2 + d^2} \tag{8b}$$

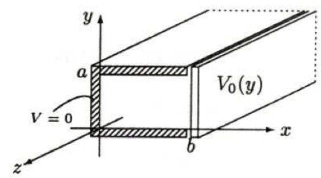
$$= -\frac{\ell \lambda d}{\pi} \left[ \frac{1}{d} \tan^{-1} \left( \frac{y}{d} \right) \right]_{y=-\infty}^{\infty} \tag{8c}$$

$$= -\frac{\ell \lambda d}{\pi} \frac{1}{d} \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right] \tag{8d}$$

$$= -\ell \lambda \tag{8e}$$

which shows the induced charge is negative the charge in the same length of the wire above the surface. Alll is happy in the world of electrostatics.

2. **Griffiths Problem 3.14:** A rectangular pipe, running parallel to the $z$-axis (from $-\infty$ to $+\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$. See the Figure below.






(a) Develop a general formula for the potential within the pipe. **Hint:** Looking at Example 3.4 in the textbook may give you some of the proper inspiration, but remember the boundary conditions on this problem are different!

(b) Find the potential explicitly for the case $V_0(y) = V_0$ (a constant).

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(a) For a two-dimensional boundary value problem in cartesian coordinates, we have shown (in class) that if we assume the solution for the potential is separable such that $V(x, y) = X(x)Y(y)$, then Laplace's Equation takes the form:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \tag{9}$$

Where the partial derivatives can be now considered ordinary derivatives (since there are no other variables than the variable we are differentiating with). and so, just like the setup for equation (3.26) in the textbook, we can split this partial differential equation into two ordinary differential equations:

$$\frac{d^2 X}{dx^2} = k^2 X; \quad \frac{d^2 Y}{dy^2} = -k^2 Y, \tag{10}$$

which has a general solution of the form

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky). \tag{11}$$

Now we need to apply the boundary conditions noted for this problem:

$$\begin{aligned} \text{(i)} \quad V(x,0) &= 0, \\ \text{(ii)} \quad V(x,a) &= 0, \\ \text{(iii)} \quad V(0,y) &= 0, \\ \text{(iv)} \quad V(b,y) &= V_0(y). \end{aligned}$$

From condition (i) we know $D = 0$. From condition (iii) we know $A = -B$. From condition (ii) we know that $ka = n\pi$ where $n$ is an integer. Using all these conditions, we can reduce equation 11 to:

$$V(x,y) = AC(e^{\frac{n\pi x}{a}} - e^{-\frac{n\pi x}{a}})\sin\left(\frac{n\pi y}{a}\right) \tag{12a}$$

$$= 2AC\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \tag{12b}$$

However, $2AC$ is just a constant and we need to consider the most general linear combination of seperable solutions, so we need to re-write equation 12a as

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \tag{13}$$

This is the general solution for the potential, but we need to determine the coefficients. This can be done by applying the last boundary condition, condition (iv), which implies:

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \tag{14}$$

Since this is a Fourier sine series, we can apply "Fourier's trick"[1]

[1] See the discussion in the textbook before equation (3.34) if you missed, or didn't follow, the discussion of Fourier's Trick in lecture.

to solve for the coefficients $C_n$ in the series:

$$\int_{y=0}^{a} V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right) \int_{y=0}^{a} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy \tag{15a}$$

$$\int_{y=0}^{a} V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy = C_n \sinh\left(\frac{n\pi b}{a}\right) \frac{a}{2} \tag{15b}$$

$$C_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_{y=0}^{a} V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy \tag{15c}$$

And so the general solution for the potential in this region is given by equation 13 where the coefficients $C_n$ can be found using equation 15ac.

(b) If $V_0(y) = V_0$, then we can fairly easily solve for the coefficients $C_n$ in the general solution since $V_0$ is a constant and so:

$$C_n = \frac{2V_0}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_{y=0}^{a} \sin\left(\frac{n\pi y}{a}\right) dy \tag{16a}$$

$$= \frac{2V_0}{a \sinh\left(\frac{n\pi b}{a}\right)} \frac{a}{n\pi} \left[-\cos\left(\frac{n\pi y}{a}\right)\right]_{y=0}^{a} \tag{16b}$$

$$= \frac{2V_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \left[-\cos n\pi + 1\right] \tag{16c}$$

$$= \frac{4V_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \text{ for } n = 1, 3, 5, \cdots \text{(odd)} \tag{16d}$$

So using this solution for the coefficients in equation 13 we have the final specific solution for the potential:

$$V(x, y) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{4V_0}{n\pi \sinh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \tag{17}$$

3. **Griffiths Problem 3.20:** Find the potential outside a *charged* metal sphere (charge $Q$, radius $R$) placed in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$. Explain clearly where you are setting the zero of potential. **BIG HINT:** This is essentially a variant of Example 3.8. Consider that potential and the fact that potentials add like scalars to finish this problem with almost no calculations!

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Since potentials add as scalars, I know the electric potential of a charged metal sphere in a uniform electric field is the same as the sum of the potential of a charged sphere plus the potential of an uncharged metal sphere in a uniform electric potential. The electric potential of an uncharged sphere in a uniform electric field is essentially Example 3.8 and culminates in equation (3.76):

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \tag{18}$$

I know that a charged metal sphere would have an electric potential outside the sphere of

$$V_{charged\ sphere}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}. \tag{19}$$

(if you didn't "know" this, you could use a variety of means to quickly show this). So summing equations 18 and 19 we find the electric potential outside a charged sphere in a uniform electric field:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \tag{20}$$

**NOTE:** A few of you were concerned about how a charged sphere could have $V = 0$ (it doesn't, that's what adding equation 19 to equation 18 handles. Yes, for equation 19 the potential was set to zero at $r = \infty$ and in equation 18 the potential was set to zero at the sphere's surface. But the only difference in equation 18 if I move the reference point elsewhere is a constant, which doesn't change the physics (which depends on differences in potential), so I'm not worrying about it.

4. **Griffiths Problem 3.34 (tweaked):** A point charge $q$ of mass $m$ is released from rest at a distance $a$ from an infinite grounded conducting plane.

   (a) Show that you can use conservation of energy to write an expression for the velocity $v$ of the charge $q$ at a height $z$ above the infinite grounded conducting plane as:

   $$v = -\sqrt{\frac{q^2}{2m}\frac{1}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{a}\right]} \tag{21}$$

   **HINT:** See section 3.2.3 of the book for an expression of the energy of a point charge above an infinite grounded conducting plane. Also note that you can safely ignore gravitational potential energy here since electric forces are much, much more powerful than gravitational forces here.

   (b) Using this expression for the total energy of the charge and the fact $v = \frac{dz}{dt}$, find an expression relating the time differential $dt$ to the positional differential $dz$. Integrate both sides of this relationship to solve for the time the charge impacts on the conducting plane, $t$. [*Answer:* $\frac{\pi a}{q}\sqrt{2\pi\epsilon_0 ma}$]

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(a) We are told this is to exploit conservation of energy, so I need to consider the forms of energy present. I know if the charge falls it has kinetic energy, but it initially starts with just potential energy. The total energy of the charge is given by the initial potential energy. The potential energy of the charge when it is a distance $z$ from the plane is given by equation 3.14:

$$U = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4z}. \tag{22}$$

Therefore the initial total energy is

$$E_0 = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4a}. \tag{23}$$

The total energy later consists of potential energy and kinetic energy, therefore:

$$E_{tot} = \frac{1}{2}mv^2 + U \tag{24}$$

for the velocity and integrate to relate the initial position to the time it takes the charge to hit the plane. The velocity is

$$\begin{aligned}
v &= \pm\sqrt{\frac{2}{m}(E_{tot} - U)} && \text{(25a)}\\
&= \pm\sqrt{\frac{2}{m}\left(-\frac{1}{4\pi\epsilon_0}\frac{q^2}{4a} - \left[-\frac{1}{4\pi\epsilon_0}\frac{q^2}{4z}\right]\right)} && \text{(25b)}\\
&= -\sqrt{\frac{q^2}{2m}\frac{1}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{a}\right]}, && \text{(25c)}
\end{aligned}$$

Since the charge is falling toward the plane, we choose the negative solution.

(b) Since $v = dz/dt$, we can separate the variables in (25a) to obtain

$$-\frac{dz}{\sqrt{\frac{q^2}{2m}\frac{1}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{a}\right]}} = dt. \tag{26}$$

Integrating both sides as the charge moves from $z = a$ to $z = 0$ in time $t = 0$ to $t = t_{impact}$ I have

$$\begin{aligned}
t_{impact} &= -\int_a^0 \left(\frac{q^2}{2m}\frac{1}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{a}\right]\right)^{-\frac{1}{2}} dz && \text{(27a)}\\
&= \int_0^a \left(\frac{q^2}{2m}\frac{1}{4\pi\epsilon_0}\left[\frac{1}{z} - \frac{1}{a}\right]\right)^{-\frac{1}{2}} dz && \text{(27b)}\\
&= \frac{2\sqrt{2\pi\epsilon_0 m}}{q}\int_0^a \left[\frac{1}{z} - \frac{1}{a}\right]^{-\frac{1}{2}} dz && \text{(27c)}\\
&= \frac{2\sqrt{2\pi\epsilon_0 m}}{q}\int_0^a \left[\frac{a - z}{az}\right]^{-\frac{1}{2}} dz && \text{(27d)}\\
&= \frac{2\sqrt{2\pi\epsilon_0 m}}{q}\int_0^a \sqrt{\frac{az}{a - z}}dz. && \text{(27e)}
\end{aligned}$$

Making the substitution $z = a\sin^2\theta$ gives $dz = 2a\cos\theta\sin\theta d\theta$ so that the integral becomes

$$t_{impact} = \frac{2\sqrt{2\pi\epsilon_0 m}}{q}\int_0^{\pi/2}\sqrt{\frac{a^2\sin^2\theta}{a(1-\sin^2\theta)}}(2a\sin\theta\cos\theta)d\theta \quad (28a)$$

$$= \frac{4a\sqrt{2a\pi\epsilon_0 m}}{q}\int_0^{\pi/2}\sin^2\theta d\theta \quad (28b)$$

$$= \frac{4a\sqrt{2a\pi\epsilon_0 m}}{q}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} \quad (28c)$$

Which results in a final expression for the time to impact of

$$t = \frac{a\pi}{q}\sqrt{2\pi am\epsilon_0}. \quad (29)$$