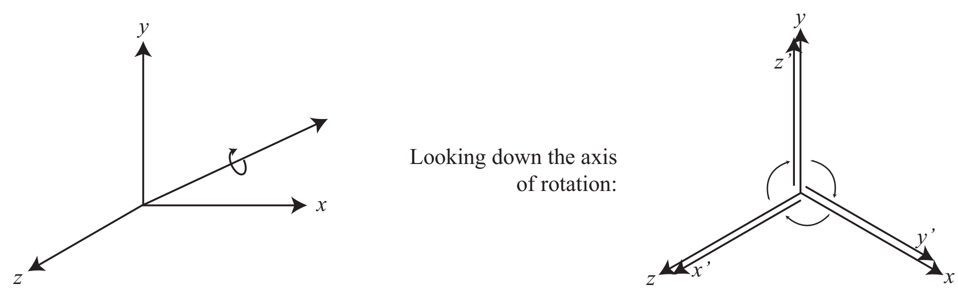
SPECIAL NOTE: I could not find a symbol that exactly matches the book's script r for the separation vector. Instead I am using the following notation: $\vec{\mathfrak{r}} = \vec{r} - \vec{r}'$, $\mathfrak{r} = |\vec{\mathfrak{r}}|$ and $\hat{\mathfrak{r}} = \vec{\mathfrak{r}}/\mathfrak{r}$.

1. **Griffiths Problem 1.9:** Find the tranformation matrix R that describes a rotation by 120° about an axis from the origin through the point (1,1,1). The rotation is clockwise as you look down the axis toward the origin.

Consider the following diagram of the situation described.



It is clear from the diagram that a 120° rotation carries the x axis onto the z $(= x')$ axis, the y axis onto the x $(= y')$, and the z axis onto the y $(= z')$ axis. Therefore $A'_x = A_z$, $A'_y = A_x$, and $A'_z = A_y$, and therefore:

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (1)$$

2. **Griffiths Problem 1.13:** Let $\vec{\mathfrak{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathfrak{r}$ be its length. Show that

   (a) $\vec{\nabla}(\mathfrak{r}^2) = 2\vec{\mathfrak{r}}$

   (b) $\vec{\nabla}(\frac{1}{\mathfrak{r}}) = -\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}$

   (c) What is the *general* formula for $\vec{\nabla}(\mathfrak{r}^n)$?

Given the definition of the sepration vector, $\vec{\mathfrak{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$, its magnitude, $\mathfrak{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$, and the definition of the del operator, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$, then we simply "plug and chug" those definitions.

(a) Since $\mathfrak{r}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$, then

$$\begin{aligned}
\vec{\nabla}(\mathfrak{r}^2) &= \frac{\partial}{\partial x}\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)\hat{x} \\
&\quad + \frac{\partial}{\partial y}\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)\hat{y} \\
&\quad + \frac{\partial}{\partial z}\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)\hat{z} && \text{(2a)} \\
&= 2(x - x')\hat{x} + 2(y - y')\hat{y} + 2(z - z')\hat{z} && \text{(2b)} \\
&= 2\vec{\mathfrak{r}} && \text{(2c)}
\end{aligned}$$

(b) Since $\frac{1}{\mathfrak{r}} = ((x-x')^2 + (y-y')^2 + (z-z')^2)^{-\frac{1}{2}}$ then

$$\begin{aligned}
\vec{\nabla}\left(\frac{1}{\mathfrak{r}}\right) &= \frac{\partial}{\partial x}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{1}{2}}\hat{x} \\
&+ \frac{\partial}{\partial y}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{1}{2}}\hat{y} \\
&+ \frac{\partial}{\partial z}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{1}{2}}\hat{z} && \text{(3a)} \\
&= -\frac{1}{2}2(x-x')\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{3}{2}}\hat{x} \\
&- \frac{1}{2}2(y-y')\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{3}{2}}\hat{y} \\
&- \frac{1}{2}2(z-z')\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{3}{2}}\hat{z} && \text{(3b)} \\
&= -\hat{\mathfrak{r}}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{3}{2}} && \text{(3c)}
\end{aligned}$$

Now, since we know the unit vector is just the vector divided by its length, we have $\hat{\mathfrak{r}} = \frac{\vec{\mathfrak{r}}}{\mathfrak{r}} = \frac{\vec{\mathfrak{r}}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$ and so

$$\begin{aligned}
\vec{\nabla}\left(\frac{1}{\mathfrak{r}}\right) &= -\hat{\mathfrak{r}}\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{3}{2}} && \text{(4a)} \\
&= -\frac{\hat{\mathfrak{r}}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}\frac{1}{(x-x')^2 + (y-y')^2 + (z-z')^2} && \text{(4b)} \\
&= -\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2} && \text{(4c)}
\end{aligned}$$

(c) To determine the *general* expression for $\vec{\nabla}(\mathfrak{r}^n)$, consider just the x coordinate. In this coordinate, the partial derivative of $\mathfrak{r}^n$ is

$$\frac{\partial}{\partial x}\mathfrak{r}^n = n\mathfrak{r}^{(n-1)}\frac{\partial \mathfrak{r}}{\partial x} \tag{5a}$$

$$= n\mathfrak{r}^{(n-1)}\left[\frac{\partial}{\partial x}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right] \tag{5b}$$

$$= n\mathfrak{r}^{(n-1)}\left[\frac{1}{2}2(x-x')\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right] \tag{5c}$$

$$= n\mathfrak{r}^{(n-1)}\left[\frac{(x-x')}{\mathfrak{r}}\right] \tag{5d}$$

You should be able to see that if we take the partial derivatives along the y and z axes, the results would have the form:

$$\frac{\partial}{\partial y}\mathfrak{r}^n = n\mathfrak{r}^{(n-1)}\left[\frac{(y-y')}{\mathfrak{r}}\right] \tag{6a}$$

$$\frac{\partial}{\partial z}\mathfrak{r}^n = n\mathfrak{r}^{(n-1)}\left[\frac{(z-z')}{\mathfrak{r}}\right] \tag{6b}$$

And putting these three derivatives together we have:

$$\vec{\nabla}(\mathfrak{r}^n) = \frac{\partial}{\partial x}\mathfrak{r}^n\hat{x} + \frac{\partial}{\partial y}\mathfrak{r}^n\hat{y} + \frac{\partial}{\partial z}\mathfrak{r}^n\hat{z} \tag{7a}$$

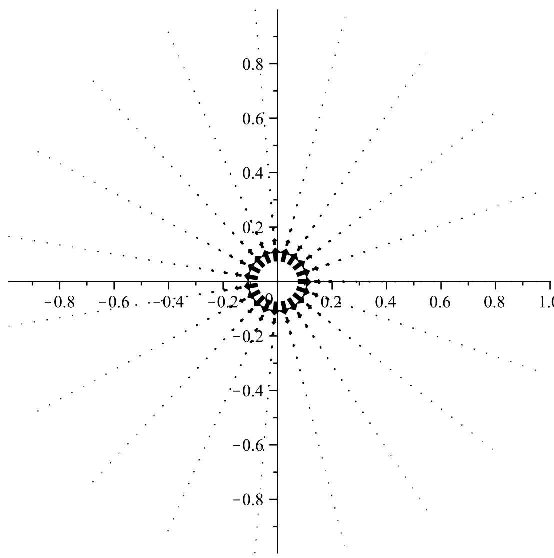
$$= n\mathfrak{r}^{(n-1)}\left[\frac{(x-x')}{\mathfrak{r}}\hat{x} + \frac{(y-y')}{\mathfrak{r}}\hat{y} + \frac{(z-z')}{\mathfrak{r}}\hat{z}\right] \tag{7b}$$

$$= n\mathfrak{r}^{(n-1)}\frac{\vec{\mathfrak{r}}}{\mathfrak{r}} \tag{7c}$$

$$= n\mathfrak{r}^{(n-1)}\hat{\mathfrak{r}} \tag{7d}$$

3. **Griffiths Problem 1.16:** Sketch the vector function $\vec{v} = \frac{\hat{r}}{r^2}$ and compute its divergence. The answer may surprise you...can you explain it?

As instructed I first sketched the vector function (using *Maple*, software available for free to any student at MSUM):



Since $\hat{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$ and $r^2 = x^2+y^2+z^2$, I can express the divergence of the function $\vec{v}$ as:

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) \tag{8}$$

The tedious part here is the derivative. Notice the x component, which via the chain rule takes the form:

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = (x^2+y^2+z^2)^{-3/2} - x\frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x) \tag{9a}$$

$$= (x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2} \tag{9b}$$

$$= r^{-3} - 3x^2r^{-5} \tag{9c}$$

The other two components take on similar forms, therefore:

$$\vec{\nabla} \cdot \vec{v} = 3r^{-3} - 3(x^2 + y^2 + z^2)r^{-5} \tag{10a}$$
$$= 3r^{-3} - 3(r^2)r^{-5} \tag{10b}$$
$$= 3r^{-3} - 3r^{-3} \tag{10c}$$
$$= 0 \tag{10d}$$

Surprised? You might be since the sketch of the vector field clearly appears to show it is diverging from the origin. So how can the divergence $\vec{\nabla} \cdot \vec{v} = 0$? The answer is that $\vec{\nabla} \cdot \vec{v} = 0$ everywhere expect at the origin. At the origin, $\vec{v}$ is actually undefined (unless you know how to divide by zero). This makes determining $\vec{\nabla} \cdot \vec{v}$ at the origin a bit tricky and part of the realm of delta functions.

4. **Griffiths Problem 1.26:** Prove the divergence of a curl is always zero. *Check* it for function $\vec{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$ (from Problem 1.15).

This is pretty straight forward, just apply the definitions of divergence and curl. The curl of the function $\vec{v}$ is:

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \tag{11a}$$
$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} \tag{11b}$$

And thus the divergence of the curl of $\vec{v}$ is:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \tag{12a}$$
$$= \left(\frac{\partial^2 v_z}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x\partial z}\right) - \left(\frac{\partial^2 v_z}{\partial y\partial x} - \frac{\partial^2 v_x}{\partial y\partial z}\right) + \left(\frac{\partial^2 v_y}{\partial z\partial x} - \frac{\partial^2 v_x}{\partial z\partial y}\right) \tag{12b}$$

By the equality of cross derivatives (*e.g.* $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$), this reduces to $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{v}\right) = 0$.

We are asked to verify this explicitly this for $\vec{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$, so we just "plug and chug":

$$\vec{\nabla} \times \vec{v_a} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 3xz^2 & -2xz \end{vmatrix} \tag{13a}$$

$$\begin{aligned} &= \left(\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2)\right)\hat{x} \\ &- \left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial z}(x^2)\right)\hat{y} \\ &+ \left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(x^2)\right)\hat{z} \end{aligned} \tag{13b}$$

$$= -6xz\hat{x} + 2z\hat{y} + 3z^2\hat{z} \tag{13c}$$

and thus:

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{v}\right) = \frac{\partial}{\partial x}(-6xz) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2) \tag{14a}$$

$$= -6z + 0 + 6z \tag{14b}$$

$$= 0 \tag{14c}$$

5. **Griffiths Problem 1.37:** Express the unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of $\hat{x}, \hat{y}, \hat{z}$ (that is, derive Eqn. 1.64). Check your answers several ways (does $\hat{r} \cdot \hat{r} = 1$, $\hat{\theta} \cdot \hat{\phi} = 0$, $\hat{r} \times \hat{\theta} = \hat{\phi}$ ...). Also work out the inverse formulas, giving $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{r}, \hat{\theta}, \hat{\phi}$.

This is the most involved problem in this problem set, mostly because there is a fair amount of tedium. Let's begin. The easiest approach to determine the spherical unit vectors is to first use the relationship between spherical and cartesian (x, y, z) coordinates from equation 1.62

$$x = r\sin\theta\cos\phi \tag{15a}$$

$$y = r\sin\theta\sin\phi \tag{15b}$$

$$z = r\cos\theta \tag{15c}$$

to note that the position vector can be written

$$\vec{r} = r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta\hat{z}. \tag{16}$$

If we take an infinitesimal step, we can represent the increase in each coordinate as:

$$d\vec{r} = \frac{\partial}{\partial r}\vec{r}dr; \quad d\vec{\theta} = \frac{\partial}{\partial \theta}\vec{r}d\theta; \quad d\vec{\phi} = \frac{\partial}{\partial \phi}\vec{r}d\phi. \tag{17}$$

And if we divide these infinitesimal vectors by their lengths, we have the coordinate unit vectors:

$$\hat{r} = \frac{d\vec{r}}{dr} = \frac{\frac{\partial}{\partial r}\vec{r}dr}{|\frac{\partial}{\partial r}\vec{r}dr|} = \frac{\frac{\partial}{\partial r}\vec{r}}{|\frac{\partial}{\partial r}\vec{r}|}; \quad \hat{\theta} = \frac{\frac{\partial}{\partial \theta}\vec{r}}{|\frac{\partial}{\partial \theta}\vec{r}|}; \quad \hat{\phi} = \frac{\frac{\partial}{\partial \phi}\vec{r}}{|\frac{\partial}{\partial \phi}\vec{r}|}. \tag{18}$$

So all we have to do is apply equation 18 by taking the appropriate derivatives of the $\vec{r}$ defined by equation 16. We should note that as is typical, the length of a vector is the square root of its components squared. As such, here begin the derivatives:

For the r direction:

$$\frac{\partial}{\partial r}\vec{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{19a}$$

$$\left|\frac{\partial}{\partial r}\vec{r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} \tag{19b}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1 \tag{19c}$$

For the θ direction:

$$\frac{\partial}{\partial \theta}\vec{r} = r\cos\theta\cos\phi\hat{x} + r\cos\theta\sin\phi\hat{y} - r\sin\theta\hat{z} \tag{20a}$$

$$\left|\frac{\partial}{\partial \theta}\vec{r}\right| = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta} = r \tag{20b}$$

$$\therefore \hat{\theta} = \frac{\frac{\partial}{\partial \theta}\vec{r}}{|\frac{\partial}{\partial \theta}\vec{r}|} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{20c}$$

For the ϕ direction:

$$\frac{\partial}{\partial \phi}\vec{r} = -r\sin\theta\sin\phi\hat{x} + r\sin\theta\cos\phi\hat{y} \tag{21a}$$

$$\left|\frac{\partial}{\partial \phi}\vec{r}\right| = \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta \tag{21b}$$

where in all the above cases I used the identity $\cos^2 x + \sin^2 x = 1$. Now we solve for the unit vectors:

$$\hat{r} = \frac{\frac{\partial}{\partial r}\vec{r}}{|\frac{\partial}{\partial r}\vec{r}|} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{22a}$$

$$\hat{\theta} = \frac{\frac{\partial}{\partial \theta}\vec{r}}{|\frac{\partial}{\partial \theta}\vec{r}|} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{22b}$$

$$\hat{\phi} = \frac{\frac{\partial}{\partial \phi}\vec{r}}{|\frac{\partial}{\partial \phi}\vec{r}|} = -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{22c}$$

At this stage, we can perform the recommended checks to see if everything is behaving as expected:

$$\hat{r}\cdot\hat{r} = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1 \ \surd \tag{23a}$$

$$\hat{\theta}\cdot\hat{\phi} = -\cos\theta\cos\phi\sin\phi + \cos\theta\cos\phi\sin\phi = 0 \ \surd \tag{23b}$$

$$\hat{r}\times\hat{\theta} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix} \tag{23c}$$

$$\begin{aligned} &= (-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)\hat{x} \\ &+ (\cos^2\theta\cos\phi + \sin^2\theta\cos\phi)\hat{y} \\ &+ (\sin\theta\cos\phi\cos\theta\sin\phi - \sin\theta\sin\phi\cos\theta\cos\phi)\hat{z} \end{aligned} \tag{23d}$$

$$= -\sin\phi\hat{x} + \cos\phi\hat{y} \tag{23e}$$

$$= \hat{\phi} \ \surd \tag{23f}$$

As a final task, we are asked to solve for the inverse relations. To do this, we'll resort to algebra (unless you want to use this same derivatives technique in reverse). First I'll try to solve for $\hat{x}$ by combining equations 22a and 22b since I see a way to eliminate the θ terms there.

$$\sin\theta\hat{r} = \sin^2\theta\cos\phi\hat{x} + \sin^2\theta\sin\phi\hat{y} + \sin\theta\cos\theta\hat{z} \tag{24a}$$

$$\cos\theta\hat{\theta} = \cos^2\theta\cos\phi\hat{x} + \cos^2\theta\sin\phi\hat{y} - \cos\theta\sin\theta\hat{z} \tag{24b}$$

$$\sin\theta\hat{r} + \cos\theta\hat{\theta} = \cos\phi\hat{x} + \sin\phi\hat{y} \tag{24c}$$

Notice that these expressions, multiplied by a sine or cosine, can be combined with equation 22c to solve for $\hat{x}$

$$\cos\phi(\sin\theta\hat{r} + \cos\theta\hat{\theta}) = \cos^2\phi\hat{x} + \cos\phi\sin\phi\hat{y} \tag{25a}$$

$$\sin\phi\hat{\phi} = -\sin^2\phi\hat{x} + \sin\phi\cos\phi\hat{y} \tag{25b}$$

$$\hat{x} = \cos\phi(\sin\theta\hat{r} + \cos\theta\hat{\theta}) - \sin\phi\hat{\phi} \tag{25c}$$

and $\hat{y}$

$$\sin\phi(\sin\theta\hat{r} + \cos\theta\hat{\theta}) = \sin\phi\cos\phi\hat{x} + \sin^2\phi\hat{y} \tag{26a}$$

$$\cos\phi\hat{\phi} = -\cos\phi\sin\phi\hat{x} + \cos^2\phi\hat{y} \tag{26b}$$

$$\hat{y} = \sin\phi(\sin\theta\hat{r} + \cos\theta\hat{\theta}) + \cos\phi\hat{\phi} \tag{26c}$$

And finally, returning to equations 22a and 22b, I can eliminate terms leaving just $\hat{z}$ by

$$\cos\theta\hat{r} = \cos\theta\sin\theta\cos\phi\hat{x} + \cos\theta\sin\theta\sin\phi\hat{y} + \cos^2\theta\hat{z} \tag{27a}$$

$$\sin\theta\hat{\theta} = \sin\theta\cos\theta\cos\phi\hat{x} + \sin\theta\cos\theta\sin\phi\hat{y} - \sin^2\theta\hat{z} \tag{27b}$$

$$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta} \tag{27c}$$