Physics 370 Midterm Exam #2 Solution Key

Spring Semester 2008

1. **(15 points TOTAL)** Answer the following conceptual problems:

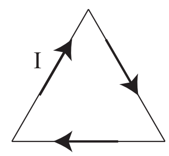
   (a) **(5 Points)** When does the method of images help solve an electrostatics problem? Put another way, if you want to solve for the electric potential/field for a given problem, what are the requirements of the corresponding images problem?

   To solve a given electrostatics problem using the method of images requires you to set up an "image problem" that has the same boundary values as the original electrostatics problem. *By the uniqueness theorem, if you find a solution to a boundary value problem that matches the boundary values, than you have found* ***the solution*** *to that boundary value problem, regardless of method used to solve the problem (including adding virtual charges to the problem such that the boundary value potentials match).*

   (b) **(5 Points)** Many particle accelerators today use a system of magnetic fields varying on short time scales to accelerate charged particles to extremely high speeds. Explain why I can confidently say that the magnetic fields were not directly responsible for the charged particle's increase in speed. That is, why I am confident the charged particle took no energy from the magnetic field, even though magnetic fields certainly played a role in helping guide the charged particles in the particle accelerator.

   We know that magnetic forces do accelerate charges, but they always act to accelerate them centripetally, that is they act at a right angle to the velocity of the charge at a given instant. As such, ***magnetic forces can do no work.*** *By the work-kinetic energy theorem, the amount of work done to an object equals its increase in kinetic energy. As such, if a charge particle is moving faster than it was before, work has been done to it... just not by magnetic forces.*

(c) **(5 Points)** Imagine for a moment that there exists a triangular current loop in the form of an equilateral triangle in the plane of this page carrying a steady current I that flows "clockwise" around the loop (as shown below). Describe an approach for determining the magnetic field (both magnitude and direction) at the center of the triangle.

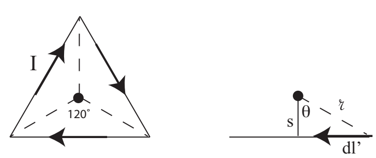


Because this isn't a calculus class, do not actually attempt a solution (although it is quite feasibe) at this time, simply state what physical principles/laws are your starting point conceptually and then describe any "tricks" or simplifications you would use to tackle the problem.

Since there are no symmetries which allow the exploitation of Ampère's law, this is a problem that requires the direct application of the Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\ell'} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}. \tag{1}$$

We can make the problem a bit more tractable by using the available symmetry to break up the problem into the problem of the magnetic field due to three equivalent straight line current segments as shown below.



Where we would then have to work out the problem of the magnetic field due to a segement of a straight line current. Using the right-hand rule on any one of the current segements allows you to state that $d\vec{\ell'} \times \hat{\mathfrak{r}}$ goes "into the page." If you state something containing the salient points above, you get full credit. To actually solve the *problem involves using the following relations:*

$$s = \mathfrak{r}\cos\theta \rightarrow \mathfrak{r} = \frac{s}{\cos\theta} \tag{2a}$$

$$d\vec{\ell'} \times \hat{\mathfrak{r}} = d\ell' \sin(\pi - \theta) \ (\textit{into of page}) = d\ell' \cos\theta \ (\textit{out of page}) \tag{2b}$$

$$d\ell' = \frac{s}{\cos^2\theta} d\theta \tag{2c}$$

to re-write the Biot-Savart law into a tractable integral.

2. **(20 points TOTAL)** Consider a cube of sides length a centered at the origin and aligned with the cartesian axes. The electrically neutral cube is composed of a dielectric that exhibits a uniform polarization $\vec{P} = P\hat{z}$.

   (a) **(6 points)** Determine the bound volume charge density ρ_b and conceptually justify its value based on the nature of bound volume charge.

   The bound volume charge density should be zero since the polarization is uniform within the cube and $\rho_b = -\vec{\nabla} \cdot \vec{P}$. Conceptually, this makes sense because bound volume charges occur only where there are divergences in the polarization which allow the ends of atomic dipoles to "pile up" at some point internally.

   (b) **(8 points)** Determine the bound surface charge density σ_b on all six sides of the cube.

   The bound surface charge density is given by the expression $\sigma_b = \vec{P} \cdot \hat{n}$. In this case, the six faces each have their own $\hat{n}$, but since $\vec{P}$ is aligned with $\hat{z}$, the only surfaces with non-zero dot product $\vec{P} \cdot \hat{n}$ are the top and bottom surfaces, so we have:

$$\sigma_b^{top} = P\hat{z} \cdot \hat{z} = P \tag{3}$$

$$\sigma_b^{bottom} = P\hat{z} \cdot (-\hat{z}) = -P \tag{4}$$

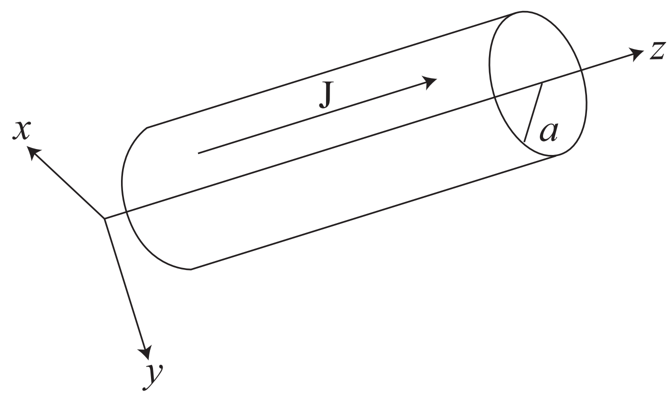
   (c) **(6 points)** What should the sum of all these bound charges be? Is this what you see?

   By conservation of charge, the sum of any bound charges should be zero. Without any bound volume charge, the only bound charge is bound surface charge on the top and bottom surfaces of the cube, which add up to a total bound charge of:

$$Q = \sigma_b^{top} a^2 + \sigma_b^{bottom} a^2 = Pa^2 - Pa^2 = 0. \tag{5}$$

   As expected, the total bound charge is zero.

3. **(30 Points TOTAL)** A very long straight conducting wire (a section of which is shown below) of radius a carries a non-uniform current density $\vec{J}(s) = \frac{Is}{\pi a^2}\hat{z}$ alng the axis of the wire where s is the radial distance in cyclindrical coordinates. [**NOTE: I majorly foobared this. The expression for volume current density was supposed to be $\vec{J}(s) = \frac{Is}{\pi a^3}\hat{z}$ [that is a^3 instead of a^2]. I had corrected this one my exam copy containing the solutions, but somehow missed it on your exam. This means if you looked carefully, you realized the results had incorrect units. However, the problem was doable as I presented it, just not physical.**]



(a) **(15 Points)** Find the magnetic field $\vec{B}$ everywhere.

This problem exhibits a clear cylindrical symmetry which lends itself to an Ampère's law approach. The amperian loop here would be co-axial with the conducting wire.

Inside the wire ($s < a$), we have

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} \tag{6a}$$

$$B(2\pi s) = \mu_0 \int \vec{J} \cdot d\vec{a}' \tag{6b}$$

$$= \mu_0 \int_{s'=0}^{s} \int_{\theta'=0}^{2\pi} \frac{Is'}{\pi a^2} s' d\theta' ds' \tag{6c}$$

$$= \mu_0 \frac{2\pi I}{\pi a^2} \int_{s'=0}^{s} s'^2 ds' \tag{6d}$$

$$= \mu_0 \frac{2Is^3}{3a^2} \tag{6e}$$

$$\vec{B}_{in} = \mu_0 \frac{Is^2}{3\pi a^2} \hat{\phi} \tag{6f}$$

Outside the wire ($s \geq a$), we have

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} \tag{7a}$$

$$B(2\pi s) = \mu_0 \int \vec{J} \cdot d\vec{a}' \tag{7b}$$

$$= \mu_0 \int_{s'=0}^{a} \int_{\theta'=0}^{2\pi} \frac{Is'}{\pi a^2} s' d\theta' ds' \tag{7c}$$

$$= \mu_0 \frac{2Ia}{3} \tag{7d}$$

$$\vec{B}_{out} = \mu_0 \frac{Ia}{3\pi s} \hat{\phi} \tag{7e}$$

(b) **(15 Points)** Compute $\vec{\nabla} \times \vec{B}$ inside and outside the wire. Do your answers make sense? Why?

NOTE: I forgot to provide the cylindrical curl expression on the equation sheet. Please bring this up when you do your exam review with me and I will give you some points back on it.

Performing the curl as requested (using cylindrical coordinates):

$$\vec{\nabla}\times\vec{B} = \left[\frac{1}{s}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial B_s}{\partial z} - \frac{\partial B_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sB_\phi) - \frac{\partial B_s}{\partial \phi}\right]\hat{z} \tag{8}$$

Noting the magnetic field here is circumferential such that $V_z = V_s = 0$*, and that* $\frac{\partial B}{\partial z} = 0$*, then the curl is much more simply expressed as:*

$$\vec{\nabla} \times \vec{B} = \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\hat{z} \tag{9}$$

So inside the wire:

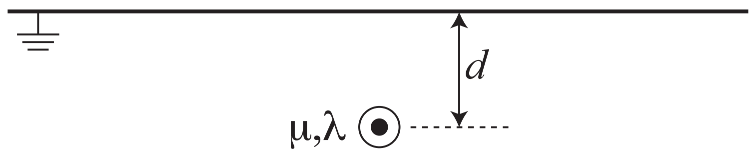
$$\vec{\nabla} \times \vec{B}_{in} = \frac{1}{s}\frac{\partial}{\partial s}(s\mu_0\frac{Is^2}{3\pi a^2})\hat{z} = \frac{1}{s}\frac{\mu_0 I}{3\pi a^2}\frac{\partial}{\partial s}(s^3)\hat{z} \tag{10a}$$

$$= \mu_0 \frac{Is}{\pi a^2}\hat{z} \tag{10b}$$

Which is just the original current density times μ_0*, expected since* $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$*. Given this expectation, would expect the curl outside the wire to be zero (since there is no current out there). Let's see:*

$$\vec{\nabla} \times \vec{B}_{out} = \frac{1}{s}\frac{\partial}{\partial s}(s\mu_0\frac{Ia}{3\pi s})\hat{z} = \frac{1}{s}\mu_0\frac{Ia}{3\pi}\frac{\partial}{\partial s}(0)\hat{z} = 0 \tag{11}$$

4. **(35 Points TOTAL)** A charge carrying wire with charge per unit length λ and mass per unit length μ lies a distance d below an "infinite" grounded conducting plane which is fixed in place. It is in equilibrium between gravity and its electrostatic attraction to the conducting plane above it.



(a) **(8 Points)** Describe the image charge distribution appropriate to this situation.

Since the conducting plane is "grounded", which means its electric potential $V = 0$, then we can create the same boundary condition at this plane by placing a charge carrying image "wire" with charge per unit length $-\lambda$ a distance d above the conducting plane. In that case, the electric potential at the location of the conducting plane would indeed be zero.

(b) **(15 Points)** Find the (upward) electrostatic force per unit length on the wire.

The upward force per unit length due to electrostatic attraction to the grounded conducting plane can be treated as being identical to upward force per unit length due to the image line charge. As such, since the image wire exhibits cylindrical symmetry, we can Gauss's law (by imagining a coaxial Guassian cylinder about the image line charge) to find the electric field due to the image line charge at the location of the read line charge.

$$\oint \vec{E} \cdot d\vec{\ell} = \frac{Q_{enc}}{\epsilon_0} \tag{12a}$$

$$E(2\pi s\ell) = \frac{-\lambda\ell}{\epsilon_0} \tag{12b}$$

$$\vec{E} = \frac{-\lambda}{2\pi s\epsilon_0}\hat{s} \tag{12c}$$

where the minus sign indicates this is an attractive force on a real (positive) line charge, that is it is an upward force. The force per unit length of the real wire is (noting the distance between the image and real line charges is $s = 2d$):

$$d\vec{F}_{elec} = dq\vec{E} = \lambda d\ell \vec{E} = \frac{-\lambda^2 d\ell}{2\pi(2d)\epsilon_0}\hat{s} \tag{13a}$$

$$\frac{d\vec{F}_{elec}}{d\ell} = \frac{\lambda^2}{4\pi d\epsilon_0} \; (upward) \tag{13b}$$

(c) **(12 Points)** Find an expression for d in terms of λ, μ, and fundamental constants.

In order for us to have equilibrium, the sum of the electrostatic force per unit length upward plus the gravitational force per unit length on the wire ($dF_{grav}/d\ell = \mu g$) downward must be zero. Therefore:

$$\frac{d\vec{F}_{elec}}{d\ell} + \frac{d\vec{F}_{grav}}{d\ell} = 0 \tag{14a}$$

$$\frac{\lambda^2}{4\pi d\epsilon_0} - \mu g = 0 \tag{14b}$$

$$d = \frac{\lambda^2}{4\pi\epsilon_0\mu g} \tag{14c}$$

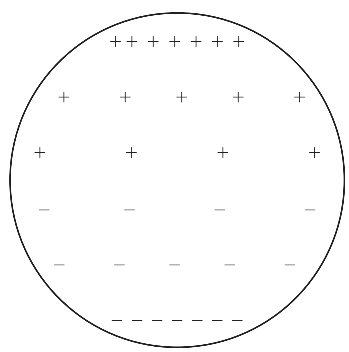
5. **(35 Points TOTAL)** A sphere of radius R, centered at the origin, carries a volume charge density

$$\rho = \frac{k}{r}\cos\theta$$

where r and θ are the usual spherical coordinates and k is a constant with the appropriate units.

(a) **(5 Points)** Sketch this charge distribution. Use your sketch to make an argument as to which term in the multipole expansion of the electric potential you expect to dominate.

A quick examination of the function $\frac{k}{r}\cos\theta$, shows us this function takes a value of $\frac{k}{r}$ at $\theta = 0$, 0 at $\theta = \pi/2$, and goes to $-\frac{k}{r}$ at $\theta = \pi$. So something like this



Where the volume charge density its maximum positive value at angles toward the "north pole". It diminishes as you approach the "equator," going to zero volume charge density at angles toward the "equator". Further "south", you get an increasingly negative volume charge density as you approach directions toward "south pole". This is clearly a dipole charge distribution with equal and opposite charges in the two hemispheres.

(b) **(10 Points)** What is the total charge of the sphere? Does this make sense given your statement about the dominant term of the multipole expansion in (a)? **HINT:** The following relationships might be useful here or later $\int \cos\theta \sin\theta d\theta = -\frac{1}{2}\cos^2\theta$ and $\int \cos^2\theta \sin\theta d\theta = -\frac{1}{3}\cos^3\theta$.

This is a simple matter of integrating the surface charge density over the entire spherical surface.

$$Q_{tot} = \int dq = \int \rho dV \tag{15a}$$

$$= \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sigma r^2 \sin\theta d\phi d\theta dr \tag{15b}$$

$$= \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} kr \cos\theta \sin\theta d\phi d\theta dr \tag{15c}$$

$$= \int_{\theta=0}^{\pi} \cos\theta \sin\theta d\theta \tag{15d}$$

$$= \pi k R \left[-\frac{1}{2}\cos^2(\pi) + -\frac{1}{2}\cos^2(0) \right] \tag{15e}$$

$$Q_{tot} = 0 \tag{15f}$$

So the total charge is zero, there is definately no monopole term for this charge distribution, as expected given my answer to (a).

(c) **(12 Points)** Find the approximate potential for points on the z axis far from and above the sphere. The first few Legendre Polynomials might be potentially useful here $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$.

Since we argued in the parts (a) and (b) that it appears the dipole expansion will dominate here, let's use it. The multipole expansion of the electric potential is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos\theta') \rho(\vec{r}') d\tau' \tag{16}$$

where the $n = 1$ term is the dipole potential:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}')d\tau' \tag{17}$$

where θ' here is angle between the $\vec{r}$ and $\vec{r}'$ vectors. If I restrict myself to the z axis, so that $\vec{r} = (r, \theta = 0)$, then θ' corresponds to the polar angle in spherical coordinates. So this is just a volume integral in spherical coordinates over a sphere for a given ρ and:

$$V_{dip}(r, \theta = 0) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\int_{r'=0}^{R} r'\cos\theta'\frac{k}{r'}\cos\theta' r'^2\sin\theta' dr'd\theta'd\phi' \tag{18a}$$

$$= \frac{k}{2\epsilon_0}\frac{1}{r^2}\int_{\theta'=0}^{\pi}\int_{r'=0}^{R} r'^2\cos^2\theta'\sin\theta' dr'd\theta' \tag{18b}$$

$$= \frac{k}{2\epsilon_0}\frac{1}{r^2}\frac{R^3}{3}\int_{\theta'=0}^{\pi}\cos^2\theta'\sin\theta' d\theta' \tag{18c}$$

$$= \frac{k}{2\epsilon_0}\frac{1}{r^2}\frac{R^3}{3}\left(-\frac{1}{3}\right)\left[\cos^3(\pi) - \cos^3(0)\right] \tag{18d}$$

$$= \frac{k}{2\epsilon_0}\frac{1}{r^2}\frac{R^3}{3}\left(\frac{2}{3}\right) \tag{18e}$$

$$= \frac{kR^3}{9\epsilon_0}\frac{1}{r^2} \tag{18f}$$

On the z axis, we can substitute $r = z$ to get:

$$V_{dip}(0, 0, z) = \frac{kR^3}{9\epsilon_0}\frac{1}{z^2}. \tag{19}$$

(d) **(8 Points)** Assuming the multipole expansion to the potential you used in part(c) is exact (and it is), find the exact electric field (magnitude *and direction*) at the point $(x, y, z) = (0, 2R, 0)$.

The electric field for a dipole is given by $\vec{E}_{dip}(\vec{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$ (where those coordinates are relative to the axis of the dipole, which in this case is the z axis) where the electric dipole is defined as $\vec{p} = \int \vec{r}'\rho d\tau'$. Notice the integral in the dipole I just did *in part (c) where I note the axis of the dipole is along the z axis...*

$$\vec{p} = \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \int_{r'=0}^{R} r' \cos\theta' \frac{k}{r'} \cos\theta' r'^2 \sin\theta' dr' d\theta' d\phi' = \frac{2}{3} \frac{kR^3}{3} 2\pi \hat{z} = \frac{4\pi k R^3}{9} \hat{z} \tag{20}$$

So the electric dipole field

$$\vec{E}_{dip}(\vec{r}) = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}) \tag{21a}$$

$$= \frac{kR^3}{9\epsilon_0 r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}). \tag{21b}$$

The position $(x, y, z) = (0, 2R, 0)$ *can be written in spherical coordinates as* $(r, \theta, \phi) = (2R, \pi/2, \pi/2)$*, therefore the electric field at this location is:*

$$\vec{E}_{dip}(0, 2R, 0) = \frac{kR^3}{9\epsilon_0 (2R)^3} (2\cos(\pi/2)\hat{r} + \sin(\pi/2)\hat{\theta}) \tag{22a}$$

$$= \frac{k}{72\epsilon_0} \hat{\theta}. \tag{22b}$$

where I will note that at this position, $\hat{\theta} = -\hat{z}$*, so we could write:*

$$\vec{E}_{dip}(0, 2R, 0) = -\frac{k}{72\epsilon_0} \hat{z}. \tag{23}$$