Physics 370 Midterm Exam #1 Solution Key

Spring Semester 2008

1. Answer the following conceptual questions:

   (a) **(10 points)** What is the electric field along the axis of an infinitely tall, uniformly charged, cylinder? *Explain your reasoning.*

   (b) **(10 points)** Consider a hollow sphere of radius R enclosing no charge but having several positive point charges near it. Is the electric field inside the sphere zero or nonzero? *Explain your reasoning.*

   (c) **(10 points)** Write an integral for the electric potential at the corner of a cube of uniform charge density with total charge q and sides of length a. **Do not attempt to solve this integral beyond the initial stage of setting it up and moving any constants or simple integrations out of the way.**

(a) *If we assume the charge distribution on the cylinder is uniform, then on the axis you are equidistant from all the charge in a given plane perpendicular to the axis, so that charge can only sum to give a field in the direction of the axis. Furthermore, since for every charge "above" you on the axis, there is a charge "below" you on the axis, these vertical fields cancel, so the net result is there is* $\boxed{\textit{zero electric field anywhere on the axis!}}$ You can also verify this result simply using Gauss' law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \tag{1a}$$

$$E(2\pi r\ell) = \frac{\rho\pi r^2\ell}{\epsilon_0} \tag{1b}$$

$$E = \frac{\rho r}{2\epsilon_0} \tag{1c}$$

$$\vec{E} = \frac{\rho r}{2\epsilon_0}\hat{r} \tag{1d}$$

which evaluated at $r = 0$ gives $\vec{E} = 0$.

(b) *There will be a* $\boxed{\textit{non-zero electric field inside the sphere}}$ *due to the external point charges. The most common incorrect answer is that some people attempt to apply Gauss' law to the sphere using a spherical Gaussian surface surrounding the sphere and since there is no enclosed charge, they argue that there is no electric field. However, the error they make is that Gauss' law is measuring the total electric flux through a closed surface* $\oint \vec{E} \cdot d\vec{a}$ *and relating that to the enclosed charge. Any electric fields due to external point charges that have field lines enter the spherical surface have then exit elsewhere, so the net electric flux is zero, which correctly suggests no enclosed charge. When we were using Gauss' law, we can only use it to determine the electric fields due to enclosed charges because any external charges cause zero net electric flux through any closed surface.*

(c) *Given the electric potential for a given charge distribution can be written*

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_{\mathcal{V}} \frac{\rho(\vec{r}')}{\mathfrak{r}} d\tau' \tag{2}$$

Without losing any generality, let me place the corner of the cube whose potential I am interested in at the origin and line up the sides of the cube in the with the axes such that the cube lies in the first octant, in other words, the cube runs from $x = 0$ *to* $x = a$, $y = 0$ *to* $y = a$, *and* $z = 0$ *to* $z = a$. *Doing this, the separation vector to any charge in the volume can be written* $\mathfrak{r} = \sqrt{x^2 + y^2 + z^2}$, *therefore for a uniform volume charge density* ρ, *the potential at the corner of the cube becomes:*

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int\limits_{x=0}^{a} \int\limits_{y=0}^{a} \int\limits_{z=0}^{a} \frac{\rho}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy \, dz && (3) \\ &= \frac{\rho}{4\pi\epsilon_0} \int\limits_{x=0}^{a} \int\limits_{y=0}^{a} \int\limits_{z=0}^{a} \frac{dx \, dy \, dz}{\sqrt{x^2 + y^2 + z^2}} && (4) \end{aligned}$$

Finally, noting that volume charge density ρ *is just charge per vol-*

ume and this charge density is uniform, then $\rho = q/a^3$, *therefore:*

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 a^3} \int\limits_{x=0}^{a} \int\limits_{y=0}^{a} \int\limits_{z=0}^{a} \frac{dx\ dy\ dz}{\sqrt{x^2 + y^2 + z^2}} \tag{5}$$

2. Consider the following two candidate electrostatic fields (electric fields due to non-moving electric charges):

$$\begin{aligned} \vec{E}_1 &= C_1\left[-y\hat{x} + x\hat{y} + 0\hat{z}\right] \qquad \text{(all space)} \\ \vec{E}_2 &= \begin{cases} C_2\left(\frac{1}{r^2} - \frac{r}{R^3}\right)\hat{r} & \text{for } 0 < r \leq R, \\ 0 & \text{for } r > R \end{cases} \end{aligned} \tag{6}$$

where C_1 and C_2 are constants that have units of $\frac{N}{m\cdot C}$ and $\frac{N\cdot m^2}{C}$ respectively, so that both vector fields have units of $\frac{N}{C}$ as appropriate for electric fields.

(a) **(10 points)** Which electric field might you see in the lab? Why wouldn't you ever observe the other?

(b) **(10 points)** For the field you determined was physically possible, identify the charge distribution ρ responsible for that field.

(a) *For a real electric field we know that $\vec{\nabla} \times \vec{E} = 0$,*[1] *so all we need to do to determine which field is physically possible is take the curl of the two fields:*

$$\begin{aligned} \vec{\nabla} \times \vec{E}_1 &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{1x} & E_{1y} & E_{1z} \end{vmatrix} \\ &= \left(\frac{\partial E_{1z}}{\partial y} - \frac{\partial E_{1y}}{\partial z}\right)\hat{x} + \left(\frac{\partial E_{1x}}{\partial z} - \frac{\partial E_{1z}}{\partial x}\right)\hat{y} + \left(\frac{\partial E_{1y}}{\partial x} - \frac{\partial E_{1x}}{\partial y}\right)\hat{z} \\ &= \left(\frac{\partial (0)}{\partial y} - \frac{\partial (x)}{\partial z}\right)\hat{x} + \left(\frac{\partial (-y)}{\partial z} - \frac{\partial (0)}{\partial x}\right)\hat{y} + \left(\frac{\partial (x)}{\partial x} - \frac{\partial (-y)}{\partial y}\right)\hat{z} \\ &= 2\hat{z} \end{aligned} \tag{7}$$

So we can conclude that $\vec{E}_1$ is not a real electric field due to static charges. To check on $\vec{E}_2$ we need to compute the curl in spherical coordinates (at least if we want to remain sane and not convert

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[1] Technically, it is only electrostatic fields that require $\vec{\nabla} \times \vec{E} = 0$, when you allow for moving charges (technically, accelerating charges), the curl of the electric field need not be zero.

the equation we were given into cartesian coordinates). First of all, clearly when $r > R$ the electric field is zero, so the curl is zero there. As such, all we need to do is check the curl for $\vec{E}_2$ when $r \leq R$:

$$\begin{aligned}\vec{\nabla} \times \vec{E}_2 &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_{2\phi}) - \frac{\partial E_{2\theta}}{\partial\phi}\right]\hat{r} \\ &+ \left[\frac{1}{r\sin\theta}\frac{\partial E_{2r}}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rE_{2\phi})\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rE_{2\theta}) - \frac{\partial E_{2r}}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

Noting that only E_{2r} is nonzero, we get:

$$\vec{\nabla} \times \vec{E}_2 = \frac{1}{r\sin\theta}\frac{\partial E_{2r}}{\partial\phi}\hat{\theta} - \frac{\partial E_{2r}}{\partial\theta}\hat{\phi}$$

and since $\vec{E}_2$ has no θ or ϕ dependence, we know:

$$\vec{\nabla} \times \vec{E}_2 = 0 \tag{8}$$

So $\boxed{\vec{E}_2 \text{ is indeed an electric field that could be produced by static charges}}$.

(b) *Using the differential version of Gauss' law, we know that:*

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \rho &= \epsilon_0 \vec{\nabla} \cdot \vec{E} \qquad (9)\end{aligned}$$

Clearly equation 9 will go to zero where there is no electric field, so there is no charge where $r > R$. For $r \leq R$, I shouldinsert $\vec{E}_2$ into equation 9 to find:

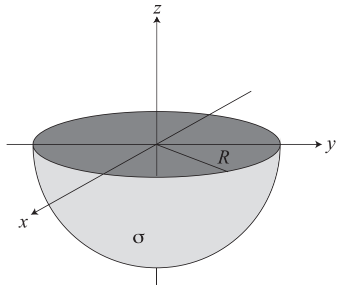
$$\begin{aligned}\rho &= \epsilon_0 \vec{\nabla} \cdot \vec{E}_2 \\ &= \epsilon_0 \vec{\nabla} \cdot \left(C_2\left(\frac{1}{r^2} - \frac{r}{R^3}\right)\hat{r}\right) \\ &= \epsilon_0\left(C_2\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) - \frac{C_2}{R^3}\vec{\nabla} \cdot (r\hat{r})\right) \qquad (10)\end{aligned}$$

Looking at the equation sheet, I know $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$ *and for the second term, I can include the* r *term from the divergence in spherical coordinates:*

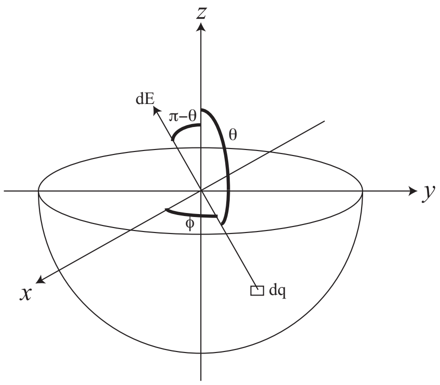
$$\begin{aligned}\rho &= \epsilon_0\left(C_2 4\pi\delta^3(\vec{r}) - \frac{C_2}{R^3}\frac{1}{r^2}\frac{\partial}{\partial r}(r^3)\right)\\ &= \epsilon_0\left(C_2 4\pi\delta^3(\vec{r}) - \frac{C_2}{R^3}\frac{1}{r^2}(3r^2)\right)\end{aligned}$$

And so finally we have the charge distribution that gave the electric field $\vec{E}_2$*:*

$$\rho = \underbrace{4\pi\epsilon_0 C_2\delta^3(\vec{r})}_{\text{“Positive” point charge at origin}} - \underbrace{\frac{3\epsilon_0 C_2}{R^3}}_{\text{unformly charged sphere with negative charge}} \qquad (11)$$



3. **(20 points)** You have a hemispherical cereal bowl with a constant surface charge density σ and a radius R. Find the electric field (magnitude and direction) at the center of the circular region (at the origin in the figure).



Consider the drawing above which shows the electric field element dE *at the origin due to a single charge element* dq *in the hemisphere. Thanks to the symmetry, I know I only need to consider the vertical component of the electric field, since all the non-vertical components will cancel out. Therefore, the electric field element takes the form:*

$$\begin{aligned} dE_z &= dE\cos(\pi-\theta) \\ &= \frac{1}{4\pi\epsilon_0}\frac{dq}{R^2}\cos(\pi-\theta) \end{aligned}$$

Now, in this case, in spherical coordinates, we can write the charge element due to the surface charge density σ *at radius* R *as* $dq = \sigma da =$

$\sigma R^2 \sin\theta d\theta d\phi$. *Therefore the electric field element takes the form:*

$$\begin{aligned} dE_z &= \frac{\sigma}{4\pi\epsilon_0}\frac{R^2\sin\theta d\theta d\phi}{R^2}\cos(\pi-\theta) \\ &= \frac{\sigma}{4\pi\epsilon_0}\sin\theta\cos(\pi-\theta)d\theta d\phi \end{aligned}$$

And now we integrate over the hemisphere, keeping in mind the limits on the integral are over the hemisphere:

$$\begin{aligned} E_z &= \frac{\sigma}{4\pi\epsilon_0}\int\limits_{\theta=\pi/2}^{\pi}\int\limits_{\phi=0}^{2\pi}\sin\theta\cos(\pi-\theta)d\theta d\phi \\ &= \frac{2\pi\sigma}{4\pi\epsilon_0}\int\limits_{\theta=\pi/2}^{\pi}\sin\theta\cos(\pi-\theta)d\theta \\ &= -\frac{\sigma}{2\epsilon_0}\int\limits_{\theta=\pi/2}^{\pi}\sin\theta\cos\theta d\theta \\ &= -\frac{\sigma}{2\epsilon_0}\left[\frac{1}{2}\sin^2\theta\right]_{\theta=\pi/2}^{\pi} \\ &= -\frac{\sigma}{2\epsilon_0}\left[0-\frac{1}{2}\right] \end{aligned}$$

Or finally, the electric field at the origin is $\boxed{\vec{E}(0,0,0) = \frac{\sigma}{4\epsilon_0}\hat{z}}$.

4. A hollow spherical shell of non-conducting material carries charge density $\rho = \frac{k}{r^2}$ in a region $a \leq r \leq b$.
   (a) **(20 points)** Find the electric field (i) inside the shell, $r < a$, (ii) within the shell, $a \leq r \leq b$, and (iii) outside the shell $r > b$.
   (b) **(5 points)** Suppose the shell instead of being an insulator was magically transformed into a conductor with the same total charge. Describe what would happen to the electric field in the three regions described in (a).
   (c) **(5 points)** Without performing any explicit calculations, can you compare the energy necessary to assemble these charge configuration in both the (a) non-conducting and (b) conducting shell cases?

(a) *The easiest way to tackle this problem is to exploit the spherical symmetry and use the integral version of Gauss' law.*

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \tag{12}$$

So we use a spherical Gaussian surface that is centered on the origin in each of the three regions such that the radial electric field is perpendicular to the surface and we can re-write equation 12 as

$$E(4\pi r^2) = \frac{Q_{enc}}{\epsilon_0} \tag{13}$$

So all we have to use equation 13 in each of the three regions.

i. For r $< a$*, there is no charge enclosed, so* $E = 0$.

ii. For $a < r < b$*, there will be charge enclosed by the Guassian surface, but since the volume charge density is not constant, I have compute the enclosed charge:*

$$Q_{enc} = \int_a^r \rho(r')d\tau' = \int\limits_{r'=a}^{r} \int\limits_{\theta'=0}^{\pi} \int\limits_{\phi'=0}^{2\pi} \frac{k}{r'^2} r'^2 \sin\theta' d\theta' d\phi' dr' = 4\pi k(r-a) \tag{14}$$

Having worked that out, I can now use equation 13 to solve for the electric field in the shell:

$$\begin{aligned} E(4\pi r^2) &= \frac{Q_{enc}}{\epsilon_0} \\ E &= \frac{1}{4\pi r^2}\frac{4\pi k(r-a)}{\epsilon_0} = \frac{k(r-a)}{\epsilon_0 r^2} \end{aligned} \tag{15}$$

iii. *For $r > b$, the total enclosed charge is just going to be the total charge in the shell, which is:*

$$Q_{enc} = \int_a^b \rho(r')d\tau' = 4\pi k(b-a) \tag{16}$$

and so the electric field out here is:

$$\begin{aligned} E(4\pi r^2) &= \frac{Q_{enc}}{\epsilon_0} \\ E &= \frac{1}{4\pi r^2}\frac{4\pi k(b-a)}{\epsilon_0} = \frac{k(b-a)}{\epsilon_0 r^2} \end{aligned} \tag{17}$$

So putting that all together, the electric field everywhere is

$$E = \begin{cases} 0, & r < a \\ \frac{k(r-a)}{\epsilon_0 r^2}, & a \leq r \leq b \\ \frac{k(b-a)}{\epsilon_0 r^2}, & r > b \end{cases} \tag{18}$$

(b) *If you replaced the shell with a conductor, then any charge on that shell would instantly move toward the outer surface. Within the conductor the electric field must be zero, and outside the conductor, the enclosed charge would be the same, so the electric field would be:*

$$E = \begin{cases} 0, & r \leq b \\ \frac{k(b-a)}{\epsilon_0 r^2}, & r > b \end{cases} \tag{19}$$

(c) *The energy is stored in the electric field when assembling a continuous charge distribution is given by the expression*

$$W = \frac{\epsilon_0}{2}\int_{all\ space} E^2 d\tau.$$

By examining equations 18 and 19, it is clear that the only difference in the electric fields of these two configuration is that in the case of the non-conductor, an electric field exists in the region $a < r < b$, but in the case of the conductor, that field is absent. Therefore, if you were going to determine the work done assembling the charges in both cases, you would need to break up the integral into:

$$W = 2\pi\epsilon_0 \left[\int\limits_{r=0}^{a} E^2 r^2 dr + \int\limits_{r=a}^{b} E^2 r^2 dr + \int\limits_{r=b}^{\infty} E^2 r^2 dr \right]$$

where in the both cases, the first integral goes to zero, the third integral is identical, the only difference is in the second integral. In the case of the conductor, since there is no E field in the conductor, this integral is zero, in the non-conductor case, it is non-zero. Therefore, ***assembling the charges in the non-conducting configuration requires more energy than in the conducting configuration.***