Physics 350 Lab 11: Nyquist Frequency

Objective: The objective of this lab is to continue to learn how to make and annotate plots in IDL and to explore the significance of the Nyquist frequency.

Background Info: In this lab you will consider discrete representations of the continuous function $\sin(kx)$. For simplicity we will assume 50 samples distributed from 0 to 2π. In the notation from lecture, $\Delta x = 2\pi/50$, and $N = 50$, and $N\Delta x = 2\pi$. The fundamental frequency is $2\pi/(N\Delta x) = 1$ for the values of N and Δx we have chosen. The Nyquist frequency is $\alpha_N = (N/2)\alpha_f$.

In this lab the functions we will sample are sines and cosines. The sampling issues come up with any function, not just trigonometric functions, but your familiarity with trigonometric functions will make it easier to interpret the results. A bit of notation that will help relate the spatial frequency α (or k in IDL) to the trigonometric functions you will plot. A cosine with wavelength λ is written $\cos(2\pi x/\lambda)$. The spatial frequency is defined to be $\alpha = 2\pi/\lambda$, so we can write the trigonometric functions in terms of the spatial frequency as

$$\cos(\alpha x) \qquad \text{[or, in IDL notation, } \cos(kx)]. \tag{1}$$

In making the plots below we need a set of points in the x-direction spaced by Δx. One way to get this is to use the commands

```
N = 50L  ; the L makes IDL treat this as a long integer
DeltaX = 2*!pi/N
x = DeltaX*findgen(N+1)  ; this actually defines N+1 points, from
                         ; 0 to 2*!pi. You can look at the values
                         ; by typing: print, x
```

All of the functions we plot will be functions of this one set of positions `x`, sampled at 50 points. It will also be convenient to have a list of positions much more finely sampled, with perhaps 10,000 points, to represent what the analytic (i.e. correct) version of the function we are plotting is. The code below generates these positions, which I call `xfine`.

```
xfine = 2*!pi/10000*findgen(10001)
```

Finally, it will be convenient to define both the fundamental frequency and the Nyquist frequency in IDL (instead of α we will use the letter k for spatial frequency, since it is a common variable choice for spatial frequency and more easily typed into IDL),

```
kf = 1.0      ; For this particular choice of DeltaX and N only.
kNyq = (N/2)*kf  ; In general
```

Now we can take a look at how well our sampling does (or doesn't) represent waves of different frequencies.

For example, we expect to be able to represent a wave with the fundamental frequency, whose wavelength is 2π, so let's try it:

```
plot, x, cos(x), TITLE='Frequency k_f'
```

So far so good. What about a high frequency wave, say at twice the Nyquist frequency? The wave should look like this:

```
plot,xfine,cos(2*kNyq*xfine)
```

but sampled over 50 points spread between 0 and 2π we get this instead:

```
plot,x,cos(2*kNyq*x),psym=2, yrange=[0,2] ; the yrange is used here to
                                          ; make sure the points
                                          ; aren't on the upper
                                          ; border of the graph
```

Looks completely different than the correct function, doesn't it?

Procedure:

1. Make plots for several different frequencies that are integer multiples of the fundamental frequency up to the Nyquist frequency. In other words, make plots of $\cos(n\alpha_f x)$ (or $\cos(nk_f x)$ in IDL notation), where n is an integer between zero and $N/2$ (recall the Nyquist frequency is $\alpha_N = (N/2)\alpha_f$.). Then answer the questions below:

(a) What is the largest n for which the cosine, sampled with 50 points, seems reasonably close to correct? Put a bit differently, if you made a graph of the cosine with 50 points connected by a line (like in `plot, x, cos(5*kf*x)`) what is the biggest n for which you could reasonably expect someone else to identify the cosine?

(b) Same as the previous part, but for a sine function, $\sin(n\alpha_f x)$. Are the answers to these parts the same?

(c) For each of the previous parts, print out one or two plots justifying your conclusions. The plots should have the horizontal and vertical axes labeled, and should have a meaningful title along the top.

2. It is remarkable just how poorly the samples of a function can represent the actual function.

   (a) Try plotting $\cos(n\alpha_f x)$ (or $\cos(nk_f x)$ in IDL notation) for a few values of n larger than $N/2$ but smaller than N. Be sure at least one of the values of n you use is $(N/2)+1$. It is helpful to look at three different graphs for each n. One should be of the sampled function (use `x` as your positions and to calculate the cosine), one should be of the correct function (use `xfine` as positions and to calculate the cosine) and one should show both. On the plot that shows both the actual function (with `plot,xfine,cos(n*kf*xfine)`) and the sampled function, plot the sampled function as squares or diamonds or some other shape point (the command `oplot` adds points or lines to the graph that is already displayed, so the command `oplot, x, cos(n*kf*x),psym=2` will add the sampled points to the plot of the actual function.

   (b) Explain why the sampled function ends up being so inaccurate. Note that your plot of both the sampled function and of the correct function should convince you that the values $\cos(n\alpha_f x)$ are being calculated correctly.

   (c) Graph the correct function $\cos((N+1)\alpha_f x)$ (or $\cos((N+1)k_f x)$ in IDL notation), by using `xfine` as your positions. You might want to make the plot window wider so that you can see the cosine more easily. Next, graph the sampled function, by using `x` as your position. If you look at the graph of the sampled function alone, what would you conclude its frequency is (remember frequency is $2\pi/\lambda$ where λ is the wavelength of the wave)?

(d) Explain why the sampled graph turned out the way it did. You may find it helpful to look at a few more examples, like $n = N + 2$ and $n = N + 3$. Your explanation here should be more than just a qualitative answer like "the sampling rate made it look bad". Derive the frequency you would expect to see in the sampled function given the correct frequency $(N + 1)\alpha_f$ and the sampling rate. As a hint, write the cosine as $\cos(Nk_f x + k_f x)$, and work out what $Nk_f x$ is for $x = q\Delta x$, where q is an integer.