Physics 350 Lab 6: Legendre Polynomials

Objective: Legendre polynomials are the solutions to Legendre's differential equation:

$$\left(1-x^2\right)\frac{d^2}{dx^2}y(x)-2x\frac{d}{dx}y(x)+\ell(\ell+1)y(x)=0.$$

It turns out this equation arises a lot in physical problems involving spherical symmetric potentials when those problems are done in spherical coordinates. As such, this differential equation pops up in classical mechanics, electricity and magnetism, spherical heat flow, quantum mechanics, and so on…

It turns out the Legendre polynomials, $P_\ell(x)$, form a complete orthonormal basis of functions. This means all the Legendre polynomials should be orthogonal to one another so their inner product with each other is:

$$\langle P_\ell(x)|P_m(x)\rangle=\int_{x=-1}^{1}P_\ell(x)P_m(x)dx=0 \qquad \text{if } \ell\neq m \tag{1}$$

and the inner product of a Legendre polynomial with itself is:

$$\langle P_\ell(x)|P_\ell(x)\rangle=\int_{x=-1}^{1}P_\ell(x)P_\ell(x)dx=\frac{2}{2\ell+1} \tag{2}$$

So a normalized Legendre polynomial function has the form:

$$\left|\sqrt{\frac{2\ell+1}{2}}P_\ell(x)\right\rangle=\sqrt{\frac{2\ell+1}{2}}P_\ell(x) \tag{3}$$

The fact that the Legendre polynomials are a complete orthonormal basis means that any well-behaved function *f(x)* on the interval $-1\leq x\leq 1$ can be written as a linear combination of Legendre polynomials – an expansion we call a "Legendre series." Specifically, for we can write

$$f(x)=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+a_3P_3(x)+\cdots=\sum_{\ell=0}^{\infty}a_\ell P_\ell(x) \tag{4}$$

where the coefficients a_0, a_1, $a_2,\ldots$ can be found by evaluating the inner product of the normalized Legendre polynomial with the function *f(x)*

$$a_\ell=\frac{(2\ell+1)}{2}\langle f(x)|P_\ell(x)\rangle=\frac{(2\ell+1)}{2}\int_{x=-1}^{1}f(x)P_\ell(x)dx \tag{5}$$

In addition to playing a bit with Legendre polynomials, you will also be getting a bit more experience writing procedures in *Maple*, since some of the functions we ask you to construct are NOT prepackaged in *Maple*.

Lab Requirements:
The supplemental *Maple* worksheet for this lab (`lab06supplement.mw`) can be found online on the class website in the Electronic Handouts area.

Procedure: The first thing to do today is to go through the supplement sheet to become more familiar with the Legendre polynomials $P_\ell(x)$

1. Verify that the Legendre polynomials are orthogonal to one another by evaluating expression (1) for several cases of ℓ and *m*.

2. Verify that the Legendre polynomials are can be normalized by evaluating expression (2) for $l = 0$, 1, and 5 and see if it matches the expression in the writeup.

3. Create a *Maple* function that calculates one of the coefficients of a Legendre series for any given function.
   a. Specifically, define a procedure called `legcoeff` so that you can type `legcoeff(cos(x),x, 3)` and have *Maple* calculate the coefficient of the $P_3(x)$ term in the Legendre series for the function $\cos(x)$. In other words, `legcoeff` executes expression (5). You can do this in one line, using the right-arrow notation, or you can write a short procedure. Either way is fine.
   b. Test `legcoeff` using the fact that first 5 coefficients of the Legendre series for the heaviside function are: $a_0 = ½$, $a_1 = ¾$, $a_2 = 0$, $a_3 = -7/16$, $a_4 = 0$, and $a_5 = 11/32$.

4. Use the previous procedure within a new procedure that will calculate all the terms in the Legendre series and add them together for any function up to some specified maximum order `nmax`. That is, write a function called `legseries` that will let you type `legseries(f,x,nmax)` and have Maple generate and then add together all of the terms in the Legendre series of expression (4) up to the `nmax`th term. Once again, you can do this either as a procedure or in one line with the right-arrow notation. **In any case, it'll be useful to look up the help on the `sum` command.**

5. Test your function from part 4 by calculating the Legendre series for the heaviside function:
$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$
up to the 10th order. Plot the Legendre series for this function, along with the function itself, over the range from $-1 \leq x \leq 1$. To appreciate how this fit could be worse or improved, also plot this for 0th, 1st, 3rd, 5th, 20th, and 40th order.