1. We are to find the Fourier transform of the function

$$f(x) = \begin{cases} x, & -1 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

and the Fourier transform $g(k)$ is given by

$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{2}$$

Putting the function f into the transform equation gives

$$g(k) = \frac{1}{2\pi} \int_{-1}^{1} x e^{-ikx} dx. \tag{3}$$

The integral can be done with integration by parts with $u = x$ and $dv = e^{-ikx}dx$, which implies that $du = dx$ and $v = -e^{-ikx}/(ik)$, so that

$$\begin{aligned} g(k) &= \frac{1}{2\pi} \left(- \frac{x e^{-ikx}}{ik} \bigg|_{-1}^{1} + \frac{1}{ik} \int_{-1}^{1} e^{-ikx} dx \right) && \text{(4a)} \\ &= \frac{1}{2\pi} \left(- \frac{e^{-ikx}}{ik} - \frac{e^{ikx}}{ik} - \frac{1}{(ik)^2} \; e^{-ikx} \Big|_{-1}^{1} \right) && \text{(4b)} \\ &= \frac{1}{2\pi} \left(- \frac{e^{ikx} + e^{-ikx}}{ik} + \frac{e^{-ikx} - e^{ikx}}{k^2} \right) && \text{(4c)} \\ &= \frac{1}{2\pi} \left(- \frac{2\cos(kx)}{ik} - \frac{2i\sin(kx)}{k^2} \right) && \text{(4d)} \\ &= \frac{ik\cos(kx) - \sin(kx)}{\pi k^2}. && \text{(4e)} \end{aligned}$$

2. We are to transform the function $f(x) = e^{-|x|}$. This can be written piecewise as

$$f(x) = \begin{cases} e^{x}, & x < 0, \\ x^{-x}, & x > 0, \end{cases} \tag{5}$$

a form that is convenient for doing the Fourier transform. The transform is

$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx \tag{6a}$$

$$= \frac{1}{2\pi} \left(\int_{-\infty}^{0} e^{(1-ik)x} dx + \int_{0}^{\infty} e^{-(1+ik)x} dx \right) \tag{6b}$$

$$= \frac{1}{2\pi} \left(\left. \frac{e^{(1-ik)x}}{1-ik} \right|_{-\infty}^{0} - \left. \frac{e^{-(1+ik)x}}{1+ik} \right|_{0}^{\infty} \right) \tag{6c}$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-ik} + \frac{1}{1+ik} \right) \tag{6d}$$

$$= \frac{1}{\pi(1+k^2)}. \tag{6e}$$

Problem 3

```
> restart;
```

Start by defining $f(x)$ and $g(k)$:

```
> f :=  (x,sigma)-> exp(-x^2/sigma^2);
```

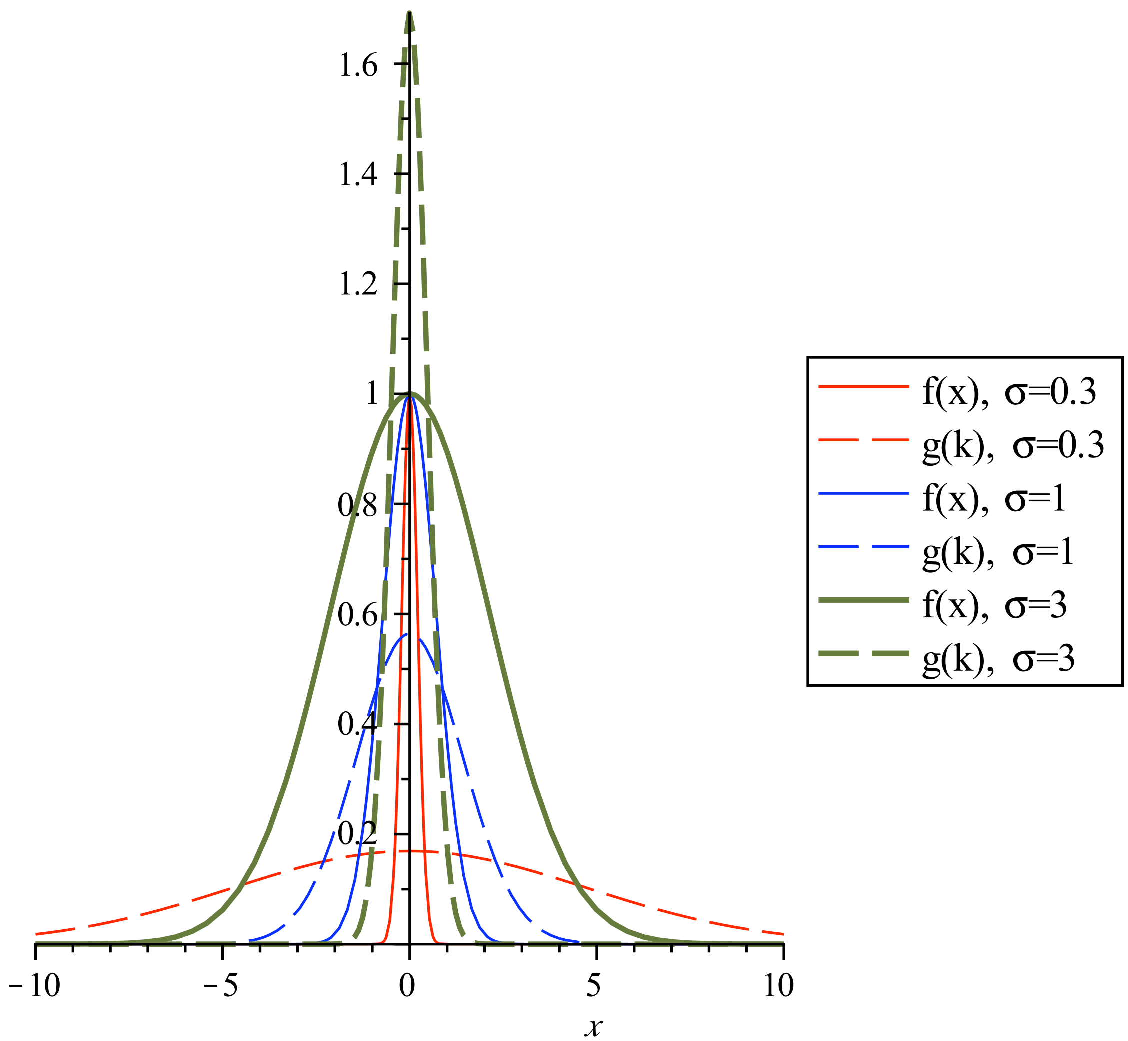
$$f := (x, \sigma) \to e^{-\frac{x^2}{\sigma^2}} \quad \textbf{(1.1)}$$

```
> g := (k,sigma)-> sigma/sqrt(Pi)*exp(-k^2*sigma^2/4);
```

$$g := (k, \sigma) \to \frac{\sigma e^{-\frac{1}{4} k^2 \sigma^2}}{\sqrt{\pi}} \quad \textbf{(1.2)}$$

Part a

```
> plot([f(x,0.3),g(x,0.3),f(x,1),g(x,1),f(x,3),g(x,3)],x,
  linestyle=[solid,dash,solid,dash,solid,dash],color=[red,red,
  blue,blue,"DarkOliveGreen","DarkOliveGreen"],legend=[typeset
  ("f(x), ",sigma,"=0.3"),typeset("g(k), ",sigma,"=
  0.3"),typeset("f(x), ",sigma,"=1"),typeset("g(k), ",sigma,"=
  1"),typeset("f(x), ",sigma,"=3"),typeset("g(k), ",sigma,"=
  3")],thickness=[1,1,1,1,2,2],legendstyle=[location=right]);
```



Part b

From the plot above it is clear that as σ increases $f(x)$ gets wider and $g(k)$ gets narrower; it turns out that the width of f is inversely proportional to width of g (where by "width" I mean "width of the peak", speaking loosely, or "full width half max" to be more precise). If I call Δx the width of f and Δk the width of g, then $\Delta x \Delta k = a$, where a is a constant. Momentum is $\hbar k$, so that $\Delta k = \frac{\Delta p}{\hbar}$ and the realtionship between the widths of f and g becomes $\Delta x \frac{\Delta p}{\hbar} = a$ so that $\Delta x \Delta p = a\hbar$, consistent with the uncertainty principle.