

1. The first is to write the right hand side in polar form. Recall that for a complex number ξ in the Cartesian form $\xi = x + iy$, the polar form is $re^{i\phi}$ where $r = \sqrt{x^2 + y^2}$ and $\tan \phi = y/x$. Here $x = 1$ and $y = 1$ so $r = \sqrt{1 + 1} = \sqrt{2}$, and $\tan \phi = 1$, so $\phi = \pi/4$, and the equation we need to solve is

$$z^4 = \sqrt{2}e^{i\pi/4}. \quad (1)$$

Because $e^{i\phi}$ is periodic with period 2π we can add any integer multiple of 2π to the argument of the exponent in (1) without changing the equation. Solving for z , and including these multiples in the exponent gives

$$z = 2^{\frac{1}{8}} e^{i(\frac{\pi}{4} + 2n\pi)/4} = \begin{cases} 2^{\frac{1}{8}} e^{i\pi/8} & n = 0, \\ 2^{\frac{1}{8}} e^{9i\pi/16} & n = 1, \\ 2^{\frac{1}{8}} e^{17i\pi/16} & n = 2, \\ 2^{\frac{1}{8}} e^{25i\pi/16} & n = 3. \end{cases} \quad (2)$$

2. Remember that the modulus of a complex number, in other words $|z|^2$, is a *real number* that is obtained one of two ways:

- either multiply z by its complex conjugate z^* (sometimes written \bar{z}), so that $|z|^2 = zz^*$, or
- use the Pythagorean theorem to write $|z|^2 = x^2 + y^2$ where x and y are the real and complex parts, respectively, of the complex number z .

The solution below shows both ways:

- Here $z = \frac{1}{3} \sin \theta + \frac{2}{3}i \cos \theta$ and $z^* = \frac{1}{3} \sin \theta - \frac{2}{3}i \cos \theta$ giving

$$|z|^2 = zz^* = \left(\frac{1}{3} \sin \theta + \frac{2}{3}i \cos \theta \right) \left(\frac{1}{3} \sin \theta - \frac{2}{3}i \cos \theta \right) = \frac{1}{9} \sin^2 \theta + \frac{4}{9} \cos^2 \theta. \quad (3)$$

- In the other method $x = (1/3) \sin \theta$ and $y = (2/3) \cos \theta$ so that

$$|z|^2 = x^2 + y^2 = \frac{1}{9} \sin^2 \theta + \frac{4}{9} \cos^2 \theta, \quad (4)$$

as above.

Problem 3: Relativity I

> **restart: with(LinearAlgebra):**

We begin by defining matrices for the Lorentz transform. I am doing this in two stages. First I define a function `relgamma` that is they from relativity:

> **relgamma:=v->1/sqrt(1-v^2/c^2);**

$$\text{relgamma} := v \rightarrow \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (3.1)$$

Next I define the Lorentz transform matrix as function of v using the function `relgamma`:

> **L := v-> Matrix([[relgamma(v), -relgamma(v)*v/c^2], [-relgamma(v)*v, relgamma(v)]]);**

$$L := v \rightarrow \text{Matrix} \left(\left[\left[\text{relgamma}(v), -\frac{\text{relgamma}(v) v}{c^2} \right], \left[-\text{relgamma}(v) v, \text{relgamma}(v) \right] \right] \right) \quad (3.2)$$

Part a

Imagine we are going to use the Lorentz transform to convert a vector $r = (t, x)$ to the rocket coordinate system $r_{\text{Rocket}} = (t_{\text{Rocket}}, x_{\text{Rocket}})$ in two steps. First we go our frame to Ichabod's frame so we multiply r the appropriate Lorentz transform $r_{\text{Ichabod}} = L_{Us \rightarrow \text{Ichabod}} r$. To complete the transform we go from Ichabod's frame to the rocket's frame, $r_{\text{Rocket}} = L_{\text{Ichabod} \rightarrow \text{Rocket}} r_{\text{Ichabod}} = L_{\text{Ichabod} \rightarrow \text{Rocket}} L_{Us \rightarrow \text{Ichabod}} r$ so the correct order for the transform from us to the rocket is $L_{\text{Ichabod} \rightarrow \text{Rocket}} L_{Us \rightarrow \text{Ichabod}}$.

Part b

For this part I will start by writing down the Lorentz transforms we need.

> **LusToRocket:=L(uptime);**

$$L_{\text{usToRocket}} := \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{\text{uptime}^2}{c^2}}} & -\frac{\text{uptime}}{\sqrt{1 - \frac{\text{uptime}^2}{c^2}}} \\ -\frac{\text{uptime}}{\sqrt{1 - \frac{\text{uptime}^2}{c^2}}} & \frac{1}{\sqrt{1 - \frac{\text{uptime}^2}{c^2}}} \end{bmatrix} \quad (3.2.1)$$

> **LusToIch:=L(v);**

$$LusToIch := \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} & -\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} c^2 \\ -\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} & \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \end{bmatrix} \quad (3.2.2)$$

> **LichToRocket := L(u);**

$$LichToRocket := \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} & -\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} c^2 \\ -\frac{u}{\sqrt{1 - \frac{u^2}{c^2}}} & \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \end{bmatrix} \quad (3.2.3)$$

The product of the transforms from Us to Ichabod then Ichabod to the rocket is

> **Lboth := LichToRocket . LusToIch;**

$$Lboth := \left[\left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}} + \frac{u v}{\sqrt{1 - \frac{u^2}{c^2}} c^2 \sqrt{1 - \frac{v^2}{c^2}}}, \right. \right. \quad (3.2.4)$$

$$\left. \left. -\frac{v}{\sqrt{1 - \frac{u^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}} c^2 - \frac{u}{\sqrt{1 - \frac{u^2}{c^2}} c^2 \sqrt{1 - \frac{v^2}{c^2}}}, \right. \right.$$

$$\left. \left. -\frac{u}{\sqrt{1 - \frac{u^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}} - \frac{v}{\sqrt{1 - \frac{u^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}}, \right. \right.$$

$$\left. \left. \frac{1}{\sqrt{1 - \frac{u^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}} + \frac{u v}{\sqrt{1 - \frac{u^2}{c^2}} c^2 \sqrt{1 - \frac{v^2}{c^2}}} \right] \right]$$

I will grab the upper left element of this set it equal to the upper left element of LusToRocket and solve for uprime:

> **solve(LusToRocket[1,1]=Lboth[1,1], uprime);**

$$\frac{c^2 (u + v)}{c^2 + u v}, -\frac{c^2 (u + v)}{c^2 + u v} \quad (3.2.5)$$

The first solution, $\frac{c^2 (u + v)}{c^2 + u v}$, is the one that gives us the standard result, $u' = \frac{(u + v)}{1 + \frac{uv}{c^2}}$.

Problem 4: Relativity II

We begin by defining the energy-momentum vector in Bob's frame,

> **Pbob:=Vector([mc^2,0]);**

$$P_{bob} := \begin{bmatrix} mc^2 \\ 0 \end{bmatrix} \quad (4.1)$$

and then us the Lorentz transform to calculate the energy-momentum vector Anne would see,

> **Panne := L(v).Pbob;**

$$P_{anne} := \begin{bmatrix} \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\ -\frac{v mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \end{bmatrix} \quad (4.2)$$

Looks pretty good; the energy Anne observes is the first component because $\mathcal{P}_A = \begin{bmatrix} E_A \\ p_A \end{bmatrix}$ so

$$E_A = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mc^2 \text{ like we had hoped. At first the momentum looks pretty far off; there}$$

seems to be a negative sign that is wrong and we are off by a factor of c^2 . The negative sign belongs there if Anne is moving to the left relative to Bob, which is the same as saying Bob is moving to the right relative to Anne. one factor of c goes away because p_A , not $p_{A'}$ is the second component of Anne's energy-momentum vector, so $p_{A'} = -m\gamma v c^2$ which apparently gives $p_A = -m\gamma v c$, which doesn't match the result we expected! Why? The root of the problem is a shortcut we took when we

introduced Lorentz transforms. I suggested making the vector $\begin{bmatrix} t \\ x \end{bmatrix}$ which doesn't make too much sense in terms of units because time and distance don't have the same units. If we fixed things up by

using the vector $\begin{bmatrix} t \\ \frac{x}{c} \end{bmatrix}$ instead the Lorentz transform would become

$$L = \begin{bmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{bmatrix}$$

which ends up taking care of the extra factor of c .

▼ Problem 5: Fun With Rotations

▼ Part (a)

The rotation matrix about the x-axis is

```
> Rx:=Matrix([[1,0,0],[0,cos(theta),-sin(theta)],[0,sin(theta),cos(theta)]]);
```

$$Rx := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad (5.1.1)$$

In the limit of small angles, $\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$, so that for small angles

```
> Rx_small:=Matrix([[1,0,0],[0,1,-theta],[0,theta,1]]);
```

$$Rx_small := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{bmatrix} \quad (5.1.2)$$

From this it we can get G_x using $G_x = \frac{(R_x - \mathbb{1})}{\theta}$, giving

```
> Gx:=Matrix([[0,0,0],[0,0,-1],[0,1,0]]);
```

$$Gx := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (5.1.3)$$

which matches Eq. (24) from the homework assignment.

▼ Part (b)

We might as well start by defining the unit vector given in the homework,

```
> Nhat:=Vector([1,1,1])/sqrt(3);
```

$$Nhat := \begin{bmatrix} \frac{1}{3} \sqrt{3} \\ \frac{1}{3} \sqrt{3} \\ \frac{1}{3} \sqrt{3} \end{bmatrix} \quad (5.2.1)$$

Next we define the generators we have not yet defined (G_x is above):

```
> Gy:=Matrix([[0,0,1],[0,0,0],[-1,0,0]]);
```

$$Gy := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (5.2.2)$$

```
> Gz:=Matrix([[0,-1,0],[1,0,0],[0,0,0]]);
```

$$G_z := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.2.3)$$

Part (i)

To enter the rotation matrix we use Eq. (25). I am breaking this into two steps. In the first I define the matrix that goes into the exponential then in the next I take the exponential.

Note that the way I get n_x , n_y , and n_z in Maple is by using `Nhat[1]`, `Nhat[2]` and `Nhat[3]`, respectively.

Note also that I am defining these functions fairly generically so that I can use the same functions in later parts of the problem when the direction \hat{N} changes.

```
> InTheExponent := Nhat -> Nhat[1]*Gx + Nhat[2]*Gy + Nhat[3]*Gz;
   InTheExponent := Nhat -> Nhat_1 Gx + Nhat_2 Gy + Nhat_3 Gz (5.2.1.1)
```

Now I take the exponential to (hopefully) get the rotation matrix

```
> R := (theta, Nhat) -> MatrixExponential(theta*InTheExponent(Nhat));
   R := (theta, Nhat) -> LinearAlgebra:-MatrixExponential(theta InTheExponent(Nhat)) (5.2.1.2)
```

For the specific case we did in lab the rotation matrix is

```
> R111 := theta -> R(theta, Nhat): R111(theta);
[[ 1/3 + 2/3 cos(theta), -1/3 sqrt(3) sin(theta) - 1/3 cos(theta) + 1/3, -1/3 cos(theta) + 1/3
  + 1/3 sqrt(3) sin(theta)],
 [ -1/3 cos(theta) + 1/3 + 1/3 sqrt(3) sin(theta), 1/3 + 2/3 cos(theta), -1/3 sqrt(3) sin(theta)
  - 1/3 cos(theta) + 1/3 ],
 [ -1/3 sqrt(3) sin(theta) - 1/3 cos(theta) + 1/3, -1/3 cos(theta) + 1/3 + 1/3 sqrt(3) sin(theta),
  1/3 + 2/3 cos(theta) ]]
```

Certainly that seems messy enough to be the right answer (and if you compare it the rotation matrix in the solutions it actually *is* the right answer).

Part (ii)

Here we are to check whether we have the right vector by applying the rotation matrix R to the same vectors we did in lab and see if we get the same result.

Below I list go through each part of the lab, starting by giving the initial vectors and the result from lab.

Lab part 3.a

Before rotation

```
> ihat := Vector ([1,0,0]); jhat := Vector([0,1,0]);  
khat := Vector([0,0,1]);
```

$$\mathbf{i}_{\text{hat}} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{j}_{\text{hat}} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{k}_{\text{hat}} := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(5.2.2.1.1.1)

▼ After rotation by 120°

```
> inew :=Vector([0,1,0]); jnew:=Vector([0,0,1]); knew:=  
Vector([1,0,0]);
```

$$\mathbf{i}_{\text{new}} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{j}_{\text{new}} := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{k}_{\text{new}} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(5.2.2.1.2.1)

▼ Using our new rotation matrix

```
> R111(2*Pi/3);
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(5.2.2.1.3.1)

```
> irot:=R111(2*Pi/3).ihat; jrot:=R111(2*Pi/3).jhat;  
krot:=R111(2*Pi/3).khat;
```

$$\mathbf{i}_{\text{rot}} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$jrot := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$krot := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(5.2.2.1.3.2)

Excellent! Our rotated vectors match the answer to the lab!

Lab part 3.b, "Now see where (1,2,4) goes after a rotation of 60° about this axis."

Before rotation

> **vec124 := Vector([1,2,4]);**

$$vec124 := \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

(5.2.2.2.1.1)

After rotation by 60°

> **vec124_new := Vector([8/3,2/3,11/3]);**

$$vec124_new := \begin{bmatrix} \frac{8}{3} \\ \frac{2}{3} \\ \frac{11}{3} \end{bmatrix}$$

(5.2.2.2.2.1)

Using our new rotation matrix

> **R11(Pi/3);**

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

(5.2.2.2.3.1)

This matrix matches the one in the lab solutions; now for the rotated vector,

> **vec124_rot := R11(Pi/3).vec124;**

$$vec124_rot := \begin{bmatrix} \frac{8}{3} \\ \frac{2}{3} \\ \frac{11}{3} \end{bmatrix}$$

(5.2.2.2.3.2)

Once again, success!

Lab part 3.c. "See where point (1,1,0) goes after a 90° rotation about this axis."

Before rotation

```
> vec110 := Vector([1,1,0]);
```

$$vec110 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(5.2.2.3.1.1)

After rotation by 90°

```
> vec110_new := Vector([2/3-1/sqrt(3), 2/3+1/sqrt(3), 2/3]);
```

$$vec110_new := \begin{bmatrix} \frac{2}{3} - \frac{1}{3}\sqrt{3} \\ \frac{2}{3} + \frac{1}{3}\sqrt{3} \\ \frac{2}{3} \end{bmatrix}$$

(5.2.2.3.2.1)

Using our new rotation matrix

```
> R111(Pi/2);
```

$$\begin{bmatrix} \frac{1}{3} & -\frac{1}{3}\sqrt{3} + \frac{1}{3} & \frac{1}{3} + \frac{1}{3}\sqrt{3} \\ \frac{1}{3} + \frac{1}{3}\sqrt{3} & \frac{1}{3} & -\frac{1}{3}\sqrt{3} + \frac{1}{3} \\ -\frac{1}{3}\sqrt{3} + \frac{1}{3} & \frac{1}{3} + \frac{1}{3}\sqrt{3} & \frac{1}{3} \end{bmatrix}$$

(5.2.2.3.3.1)

```
> vec110_rot := R111(Pi/2).vec110;
```

$$vec110_rot := \begin{bmatrix} \frac{2}{3} - \frac{1}{3}\sqrt{3} \\ \frac{2}{3} + \frac{1}{3}\sqrt{3} \\ \frac{2}{3} \end{bmatrix}$$

(5.2.2.3.3.2)

Once again, success!

Part (c)

Finding the matrix that rotates by angle θ about the axis (1/3, 2/3, 2/3) is straightforward using the definition in part (a) above,

```
> axis := Vector([1/3, 2/3, 2/3]);
```

$$axis := \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \quad (5.3.1)$$

```
> Rc:=theta->R(theta,axis): Rc(theta);
```

$$\left[\left[\frac{1}{9} + \frac{8}{9} \cos(\theta), -\frac{2}{3} \sin(\theta) - \frac{2}{9} \cos(\theta) + \frac{2}{9}, -\frac{2}{9} \cos(\theta) + \frac{2}{9} + \frac{2}{3} \sin(\theta) \right. \right. \quad (5.3.2)$$

$$\left. \left. \begin{array}{l} \left[-\frac{2}{9} \cos(\theta) + \frac{2}{9} + \frac{2}{3} \sin(\theta), \frac{4}{9} + \frac{5}{9} \cos(\theta), -\frac{4}{9} \cos(\theta) + \frac{4}{9} \right. \right. \\ \left. \left. - \frac{1}{3} \sin(\theta) \right], \right. \\ \left[-\frac{2}{3} \sin(\theta) - \frac{2}{9} \cos(\theta) + \frac{2}{9}, -\frac{4}{9} \cos(\theta) + \frac{4}{9} + \frac{1}{3} \sin(\theta), \frac{4}{9} \right. \\ \left. \left. + \frac{5}{9} \cos(\theta) \right] \right] \end{array} \right]$$

Wow, messy. Let's check using the vector given in the lab.

```
> vBefore:=Vector([2,1,-2]);
```

$$vBefore := \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad (5.3.3)$$

```
> vAfter:=Rc(Pi/2).vBefore;
```

$$vAfter := \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \quad (5.3.4)$$

Excellent! It seems like we have the right rotation.