Physics 350 Problem Set 7 (Spring Semester 2009)
Due Thu., March 5 at 4:30PM

First a couple fairly random questions...

1. Find all solutions to the equation $z^4 = 1 + i$ BY HAND and then check your results with Maple. You had some problems like this on previous homework...dig them out if you need to!

2. Calculate $|z|^2$ BY HAND for $z = \frac{1}{3}\sin\theta + \frac{2}{3}i\cos\theta$.

   Now some problems related to matrices and transformations. This is a really cool and useful topic, though as you saw in lab working with matrices can sometimes be a little painful. We hope these problems will help demonstrate some of the power of matrices.

3. **Special relativity I**: The first page or so is explanation. The actual question to answer come later. We wrote down the Lorentz transform in class. It was

$$L(v) = \begin{pmatrix} \gamma & -\gamma\frac{v}{c^2} \\ -\gamma v & \gamma, \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & -\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}} \\ -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} \tag{1}$$

   where

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}; \tag{2}$$

   note that both L and γ depend on velocity. Writing the Lorentz transform as a matrix provides a straightforward way to derive the relativistic velocity addition formula, a way that is made much easier by Maple.

   We will start with some notation. Suppose there are two observers, you and your friend Ichabod. Ichabod is moving with velocity v to the right according to you. Ichabod observes a rocket and measures its velocity to be u. The relativistic velocity addition formula lets you figure what velocity u' you would measure for the rocket,

$$u' = \frac{u+v}{1+\frac{uv}{c^2}}. \tag{3}$$

One more time, for emphasis, the meaning of each variable in this equation is:

- v is the speed you observe Ichabod to move.
- u is the velocity Ichabod observes the rocket to move.
- u' is the velocity you observe the rocket to move.

To derive the velocity addition formula we will think about Lorentz transforms. The Lorentz transform from our frame to the frame of the rocket can be written two way. One way is

$$L_{\text{us}\to\text{rocket}} = L(u') = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} & -\frac{u'}{c^2\sqrt{1-\frac{u'^2}{c^2}}} \\ -\frac{u'}{\sqrt{1-\frac{u'^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} \end{pmatrix}. \tag{4}$$

The other way to write the transform is

$$\begin{aligned} L_{\text{us}\to\text{rocket}} &= L_{\text{Ichabod}\to\text{rocket}} L_{\text{us}\to\text{Ichabod}} && \text{(5a)} \\ &= L(u)L(v) && \text{(5b)} \\ &= \begin{pmatrix} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} & -\frac{u}{c^2\sqrt{1-\frac{u^2}{c^2}}} \\ -\frac{u}{\sqrt{1-\frac{u^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & -\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}} \\ -\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} && \text{(5c)} \end{aligned}$$

So how do you get u' out of all of this? Set (5c) equal to (4) and solve for u', a task that would be messy were it not for Maple. You don't actually have to set the two matrices $L(u')$ and $L(u)L(v)$ equal to each other. You can pick just a single entry, say the row 1, column 1 entry, set those equal and solve for u'

The questions to answer:

(a) Explain conceptually why the order in (5a) is $L_{\text{Ichabod}\to\text{rocket}} L_{\text{us}\to\text{Ichabod}}$ and not $L_{\text{us}\to\text{Ichabod}} L_{\text{Ichabod}\to\text{rocket}}$.

(b) Use Maple to solve for u' in terms of u, v and c. Technical hints:

  - Start your worksheet by including the LinearAlgebra package by doing `with(LinearAlgebra)`: This is slightly different than what you did in lab.

- Define your matrices using the command `Matrix` rather than the `matrix` you used in lab.
- Multiply matrices by using a period. If `A` and `B` are matrices in Maple then `A.B` is the product of `A` and `B`.
- Rather than writing three different matrices by hand define a function called `L` that takes a velocity as its argument and returns the matrix (1).

4. **Special Relativity II**: In class we talked about a time-space vector

$$\begin{pmatrix} t \\ x \end{pmatrix}. \tag{6}$$

It turns out you can think of energy and momentum as being two components of a vector also. We will call this energy-momentum vector $\vec{P}$, and define it by

$$\vec{P} \equiv \begin{pmatrix} E \\ pc \end{pmatrix}, \tag{7}$$

where E is energy, p is momentum and c is the speed of light. The nice thing about this energy-momentum vector is that you can use the Lorentz transform to figure out what energy and momentum observers moving at different velocities will measure.

Suppose Bob observes a proton and measures its energy and momentum to be E_B and p_B, respectively. Then the energy-momentum vector for this particle according to Bob is

$$P_B = \begin{pmatrix} E_B \\ p_B c \end{pmatrix}. \tag{8}$$

If Annie is moving at velocity v relative to Bob then the energy-momentum vector than Annie will observe for this particle is

$$P_A = L(v) P_B, \tag{9}$$

where $L(v)$ is the Lorentz transform given in Eq. (1).
The stuff for you to do:
Derive the relativistic energy and momentum formulas ($E = \gamma mc^2$ and $p = \gamma mv$) using the Lorentz transform. Get started by assuming Bob measures the energy of a particle of mass m that is not moving

relative to Bob and finds its energy and momentum to be $E_B = mc^2$ and $p_B = 0$, then use (9) to figure out the energy E_A and momentum p_A that Annie will measure for the particle.

5. Next we will have some fun with rotations....OK, now that you have stopped either laughing or crying really hard at hearing both "fun" and "rotation" in the same sentence, read on. Again, there will be a page or so of explanation followed by things for you to do. You will not be able to do those things without reading this explanation, though, so grab some tea or coffee, settle into a comfy chair and start reading.

   In lab this past week you constructed a rotation matrix about an axis pointing in the direction $(1, 1, 1)$. This problem will take you through a very different way of constructing this matrix. We begin with the rotations about each of the axes,

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{10}$$

   which rotates a vector *counterclockwise* by angle θ about the x-axis,

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{11}$$

   which rotates *counterclockwise* by angle θ about the y-axis, and

$$R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{12}$$

   which rotates a vector *counterclockwise* by angle θ about the z-axis.

   Our approach is to start by thinking about small rotations, and I'll focus on rotations about the z-axis. If the rotation angle is small (I'll call it $\Delta\theta$) then a rotation about the z-axis is

$$R_z(\Delta\theta) = \begin{pmatrix} 1 & -\Delta\theta & 0 \\ \Delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

which we could write as

$$R_z(\Delta\theta) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & -\Delta\theta & 0\\ \Delta\theta & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{14}$$

$$\equiv \mathbf{1} + \Delta\theta G_z, \tag{15}$$

where **1** is the identity matrix and the matrix G_z, called the generator of rotations about the z-axis, is defined to be

$$G_z \equiv \frac{1}{\Delta\theta}(R_z(\Delta\theta) - \mathbf{1}) = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{16}$$

How is this useful? It turns out we can relate this rotation by a small angle to the exact rotation by a large angle θ, given in (12), by using a result from Calculus I,

$$\lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N = e^x. \tag{17}$$

Imagine breaking the rotation through angle θ into N smaller steps, each of size $\Delta\theta = \theta/N$. Then

$$R(\theta) = \overbrace{R(\Delta\theta)\cdot R(\Delta\theta)\cdots R(\Delta\theta)}^{N \text{ times}} \tag{18}$$

$$= (R(\Delta\theta))^N \tag{19}$$

$$= (\mathbf{1} + \Delta\theta G_z)^N \tag{20}$$

$$= \left(\mathbf{1} + \frac{\theta}{N}G_z\right)^N. \tag{21}$$

This bears an uncanny resemblance to (17)...we'll make the leap of faith that there is such a thing as the exponential of a matrix and conclude that

$$R(\theta) = \lim_{N\to\infty}(R(\Delta\theta))^N = \lim_{N\to\infty}\left(\mathbf{1} + \frac{\theta}{N}G_z\right)^N = e^{\theta G_z}; \tag{22}$$

for those interested in what the matrix of an exponential means there is some additional info at the very end of this problem set.

By now you may be wondering why any of this is useful. There are two answers, one mathematical and one physical. The physical one is related to quantum mechanics and will be described in a little bit. The mathematical reason is that writing rotations as an exponential simplifies writing rotations in some cases. For example, we could show, using an argument just like the one we used above for the z-axis, that we can write

$$R_x(\theta) = e^{\theta G_x} \quad \text{and} \quad R_y(\theta) = e^{\theta G_y}, \tag{23}$$

where the generators G are obtained by looking at small rotations about the appropriate axis. It turns out that

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad G_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{24}$$

Now the fun part. Suppose you want to figure out the matrix for a rotation by an angle θ about an axis that points in the direction $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$. It turns out that rotation is given by

$$R(\theta, \hat{n}) = \exp((n_xG_x + n_yG_y + n_zG_z)\theta) = \exp(\hat{n} \cdot \vec{G}\theta), \tag{25}$$

a formula it is easy to enter into Maple.

A few words about applications of this stuff to physics. One application of this is in quantum mechanics. All particles have intrinsic angular momentum, called spin. Think of elementary particles like electrons as spinning on their axis. Different kinds of particles have different spins and there are some particles with what is called spin 1. These particles can have one of three different values for the z-component of their angular momentum: +1, 0, or -1. These three states are analogous to the vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, and it turns out the quantum mechanical operator that describes the rotation of one of these particles about the x-axis is $\exp -i\theta J_x/\hbar$, where J_x is a matrix (for those in quantum, it is the angular momentum operator).

You may still be wondering how this is physics. Dr. Shastri's NMR lab uses a magnetic field to cause the protons in hydrogen to rotate and that motion can be described using these types of rotation operators. As the folks in quantum this semester are learning, you can even write down the fundamental equation of quantum mechanics, Schroediniger's equation, in terms of the exponential of a matrix.

In classical mechanics this sort of approach to rotations is useful for describing the motion of spinning objects.

The stuff for you to do:

(a) Show that the expression for G_x given above is correct by starting with R_x and looking at the small angle limit. Note that G_x can be written as $R_x - \mathbf{1}$, and a similar expression works for the other directions.

(b) Use the unit vector

$$\hat{n} = \frac{1}{\sqrt{3}}(1,1,1) \tag{26}$$

in the formula (25) to:

  i. find the same matrix you did in lab...the matrix that rotates by angle θ and the direction $(1,1,1)$. This should not take very many lines of Maple; you need to define the generators, then implement the formula. You do need to know that the way to take the exponential of a matrix `A` in Maple is `MatrixExponential(A)`.

  ii. Verify that you have the correct matrix by letting your matrix operate on the same vectors as in Lab 7, Question 3.b and making sure you get the same result as in lab (check the lab solutions online).

(c) Find the rotation matrix that rotates vectors by angle θ about the axis $(1/3, 2/3, 2/3)$. Hint: You will know you have the right answer if rotating the vector $(2,1,-2)$ by $\Pi/2$ gives $(-2,2,-1)$.