Physics 350 Problem Set 6 (Spring Semester 2009)
Due Thu., February 26 at 4:30PM

1. You may have noticed the odd-numbered Legendre functions (P_1, P_3, P_5, and so on) are odd (antisymmetric) functions and the even-numbered Legendre functions (P_0, P_2, P_4, and so on) are even (symmetric) functions, in the sense of odd and even defined in the last set of homework problems. For each of the functions below explain whether you think its expansion in terms of Legendre polynomials will contain only odd-numbered terms, only even numbered terms, or a mix of odd and even terms and **explain your reasoning**. Note that this is very similar to part of what you did in Lab 5 when you looked at Fourier sine and cosine expansions. Feel free to graph these functions if you think it will be helpful in determining the answer, but you do not have to turn in those graphs.

   (a) $f(x) = \sin(\pi x)$
   (b) $f(x) = x^2 e^{-x^2}$
   (c) $f(x) = xe^{-x}$
   (d) $f(x) = (x^2 - 1)^2$

2. **(Worth 30 points)** We now have three very different ways of approximating a function defined on the interval $x = (-1, 1)$. We can find the Taylor series approximation to the function about the point $x = 0$, we could expand the function as a sum of Legendre polynomials, or we could perform a Fourier series expansion of the function.

   In this problem you will expand the same function these three different ways and comment on which approximation seems to be most accurate for the fewest terms in the expansion. In other words we want to get a handle on which approximation method is the "best" in the sense that it accurately represents the function of interest with the fewest terms.

   To be "fair" to each of the methods I've chosen a function that is neither a simple polynomial nor a simple trigonometric function. Consider the function $f(x) = x^2 e^{-x^2} \cos(\sqrt{x})$. In the parts (a) through (c) below you will approximate this function in different ways. Include as many terms as you need to that the function is reasonably well approximated

by the expansion over the range $x = (-1, 1)$. By "reasonably well" I don't have anything specific in mind (like requiring that the average difference between the function and its approximation is less than 0.1); if you plot the function and the approximation and they match pretty well that is good enough. The Maple code you need for this problem is on the course website on the "Assignments" page. Its filename is `Homework06_supplement.mw`.

(a) Plot, *on the same graph*, the Taylor series expansion of $f(x)$ about 0 for at least two different numbers of terms and $f(x)$ itself. Decide what the smallest number of terms is that you need in the Taylor series to approximate $f(x)$ reasonably well over the range $x = -1$ to $x = 1$.

(b) Plot, *on the same graph*, the function $f(x)$ and the Fourier series representation of $f(x)$ for the range $x = (-1, 1)$ including at least two different numbers of terms. Decide what the smallest number of terms you need to match $f(x)$ reasonably well is.

(c) Plot, *on the same graph*, the function $f(x)$ and the Legendre polynomial series of $f(x)$ for at least two different numbers of terms. Decide what the smallest number of terms you need to match $f(x)$ reasonably well is.

(d) Plot, *on the same graph* , $f(x)$, and the "best" of each of the series you found with range $x = -1$ to $x = 1$. In other words, you want $f(x)$ and the best Taylor series, the best Fourier series, and the best Legendre series on one graph. Which of the approximations needed the fewest terms to provide a good fit? Or are they all about the same? Comment on your results.

(e) Now plot the same functions as in the previous part, but for a range in x of -2 to 2 and answer these questions:

   i. Why don't the series approximate $f(x)$ as well outside the range -1 to 1?

   ii. Which, if any, of the series would fit $f(x)$ over the range -2 to 2 if you added more terms? Which would not, no matter how many terms you added?

(f) What kind of function would Fourier series be best at representing? What about Taylor series or Legendre series?