1. (a) We are to find the solution to the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = 0. \tag{1}$$

Trying a solution of the form $x = e^{at}$ gives an equation for a,

$$a^2 + a - 6 = 0. \tag{2}$$

The solutions to this are $a = 2$ and $a = -3$ so the general solution to the differential equation is

$$x(t) = C_1 e^{2t} + C_2 e^{-3t}, \tag{3}$$

where C_1 and C_2 are constants of integration determined by the initial conditions.

(b) We are told that $x(0) = 2$ and $v(0) = 0$. From our solution (3) we also know that

$$x(0) = C_1 + C_2, \tag{4}$$

so that

$$2 = C_1 + C_2. \tag{5}$$

The velocity is

$$v(t) = \frac{dx}{dt} = 2C_1 e^{2t} - 3C_2 e^{-3t}, \tag{6}$$

so that

$$v(0) = 2C_1 - 3C_2 \tag{7}$$

and

$$0 = 2C_1 - 3C_2 \qquad \Longrightarrow \qquad C_2 = \frac{2}{3}C_1. \tag{8}$$

Putting this result for C_2 into (5) gives

$$2 = C_1 + \frac{2}{3}C_1 = \frac{5}{3}C_1 \qquad \Longrightarrow \qquad C_1 = \frac{6}{5} \qquad \Longrightarrow \qquad C_2 = \frac{2}{3}C_1 = \frac{4}{5}. \tag{9}$$

2. We are to find the solution to the differential equation

$$9\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + x = 0. \tag{10}$$

Trying a solution of the form $x = e^{at}$ gives an equation for a,

$$9a^2 + 6a + 1 = 0, \qquad \Longrightarrow \qquad (3a+1)^2 = 0, \tag{11}$$

which has solution $a = -1/3$.When there are two solutions for a, like in the previous problem, we get two solutions to our differential equation, one for each value of a. Here we only have one solution even though the mathematicians say that there are always two solutions to a second order differential equation. The answer is that the second solution is not a simple exponential; instead the two solutions are $e^{-t/3}$ and $te^{-t/3}$, so that

$$x(t) = C_1 e^{-t/3} + C_2 t e^{-t/3} = (C_1 + C_2 t)e^{-t/3}. \tag{12}$$

We are given the initial conditions $x(0) = 12$ and $v(0) = 5$. From the solution (12) the initial position is $x(0) = C_1$, so $C_1 = 12$. The velocity is

$$v(t) = \frac{dx}{dt} = -\frac{C_1}{3}e^{-t/3} + C_2 e^{-t/3} - \frac{C_2 t}{3}e^{-t/3}, \tag{13}$$

so that

$$v(0) = -\frac{C_1}{3} + C_2 = 5 \qquad \Longrightarrow \qquad C_2 = 5 + \frac{C_1}{3} = 9, \tag{14}$$

so that the solution in this case is

$$x(t) = (12 + 9t)e^{-t/3}. \tag{15}$$

3. We consider the differential equation

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_N^2 x = f\cos(\omega_D t). \tag{16}$$

whose particular solution is $x(t) = Ae^{i\omega_D t}$. We are to determine A. Substituting x into the differential equation gives

$$\begin{aligned} &-\omega_D^2 A e^{i\omega_D t} + 2\beta(i\omega_D)Ae^{i\omega_D t} + \omega_N^2 A e^{i\omega_D t} = 0 \\ &\qquad \Rightarrow A e^{i\omega_D t}(\omega_N^2 - \omega_D^2 + 2i\beta\omega_D) = 0. \end{aligned} \tag{17}$$

The exponential factor divides out so solving for A gives

$$A = \frac{1}{(\omega_N^2 - \omega_D^2) + i(2\beta\omega_D)}. \tag{18}$$

Next we are to write this in polar form. A is a fraction of the form $1/(x+iy)$ and the way we have written that sort of fraction in polar form has been to multiply by $(x-iy)/(x-iy)$. We do the same thing here, with $x = (\omega_N^2 - \omega_D^2)$ and $y = 2\beta\omega_D$, to get

$$A = \frac{1}{(\omega_N^2 - \omega_D^2) + i(2\beta\omega_D)} \left(\frac{(\omega_N^2 - \omega_D^2) - i(2\beta\omega_D)}{(\omega_N^2 - \omega_D^2) - i(2\beta\omega_D)} \right) = \frac{(\omega_N^2 - \omega_D^2) - i(2\beta\omega_D)}{(\omega_N^2 - \omega_D^2)^2 + (2\beta\omega_D)^2}. \tag{19}$$

This is A written in $x+iy$ form. The real part of A is

$$\Re(A) = \frac{(\omega_N^2 - \omega_D^2)}{(\omega_N^2 - \omega_D^2)^2 + (2\beta\omega_D)^2} \tag{20}$$

and its imaginary part is

$$\Im(A) = \frac{-2\beta\omega_D}{(\omega_N^2 - \omega_D^2)^2 + (2\beta\omega_D)^2}. \tag{21}$$

Recall that polar form of a number $x+iy$ is $re^{i\delta}$ where $r = \sqrt{x^2+y^2}$ and $\delta = \arctan(y/x)$. With these, the magnitude of A is

$$|A| = \left(\frac{(\omega_N^2 - \omega_D^2)^2}{((\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2)^2} + \frac{4\beta^2\omega_D^2}{((\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2)^2} \right)^{1/2} \tag{22a}$$

$$= \left(\frac{(\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2}{((\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2)^2} \right)^{1/2} \tag{22b}$$

$$= \left(\frac{1}{(\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2} \right)^{1/2} \tag{22c}$$

$$= \frac{1}{\sqrt{(\omega_N^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2}} \tag{22d}$$

Getting the phase angle δ is a bit easier. It is simply $\delta = \tan^{-1}(-2\beta\omega_D/(\omega_N^2 - \omega_D^2))$.

With these results in hand we can make a bit more sense of last week's lab. The amplitude $|A|$ is biggest when its denominator is smallest, which happens when $\omega_D \approx \omega_N$ (the actual value of ω_D that maximizes the amplitude is called the resonant frequency; you would calculate it by taking the derivative of $|A|$ with respect to ω_D and setting the derivative equal to zero, then solving for ω_D, a process that is a touch messy). When $\omega_D < \omega_N$ the angle δ will be positive; if the driving frequency is very small compared to the natural frequency then the angle is roughly 0 and the oscillator vibrates in phase with the driving force. When the driving frequency ω_D is very large compared to ω_N then the angle δ will approach π and the oscillator will be exactly out of phase with the driving force, which matches the behavior we saw in lab. When the driving frequency is near the natural frequency the angle $\delta \approx \pi/2$.

```
> restart; with(plots):
```

Below we define the differential equation we are to solve. It differs from what you did in lab only by the addition of the extra driving term on the right hand side.

```
> odeb := (diff(y(eta), eta$2))+bb*(diff(y(eta), eta))+(2*Pi)^2*y
  (eta) = f*cos(omgT*eta)+f*cos(1.5*omgT*eta);
```

$$odeb := \frac{d^2}{d\eta^2} y(\eta) + bb \left(\frac{d}{d\eta} y(\eta) \right) + 4\pi^2 y(\eta) = f\cos(omgT\,\eta) + f\cos(1.5\, omgT\,\eta) \qquad \textbf{(1)}$$

```
> icsb := y(0) = 0, D(y)(0) = 0:
> dsolve({odeb, icsb}, y(eta)):
```

The solution, simplified, is below. Even simplified it is messy.

```
> soly := rhs(%):
> simplify(soly):
```

We will use the same damping and driving amplitude that we did in lab.

```
> bb := (2*Pi)*(1/20): f:=26.48:
```

For making the plots it is convenient to make the solution a function of **omgT**.

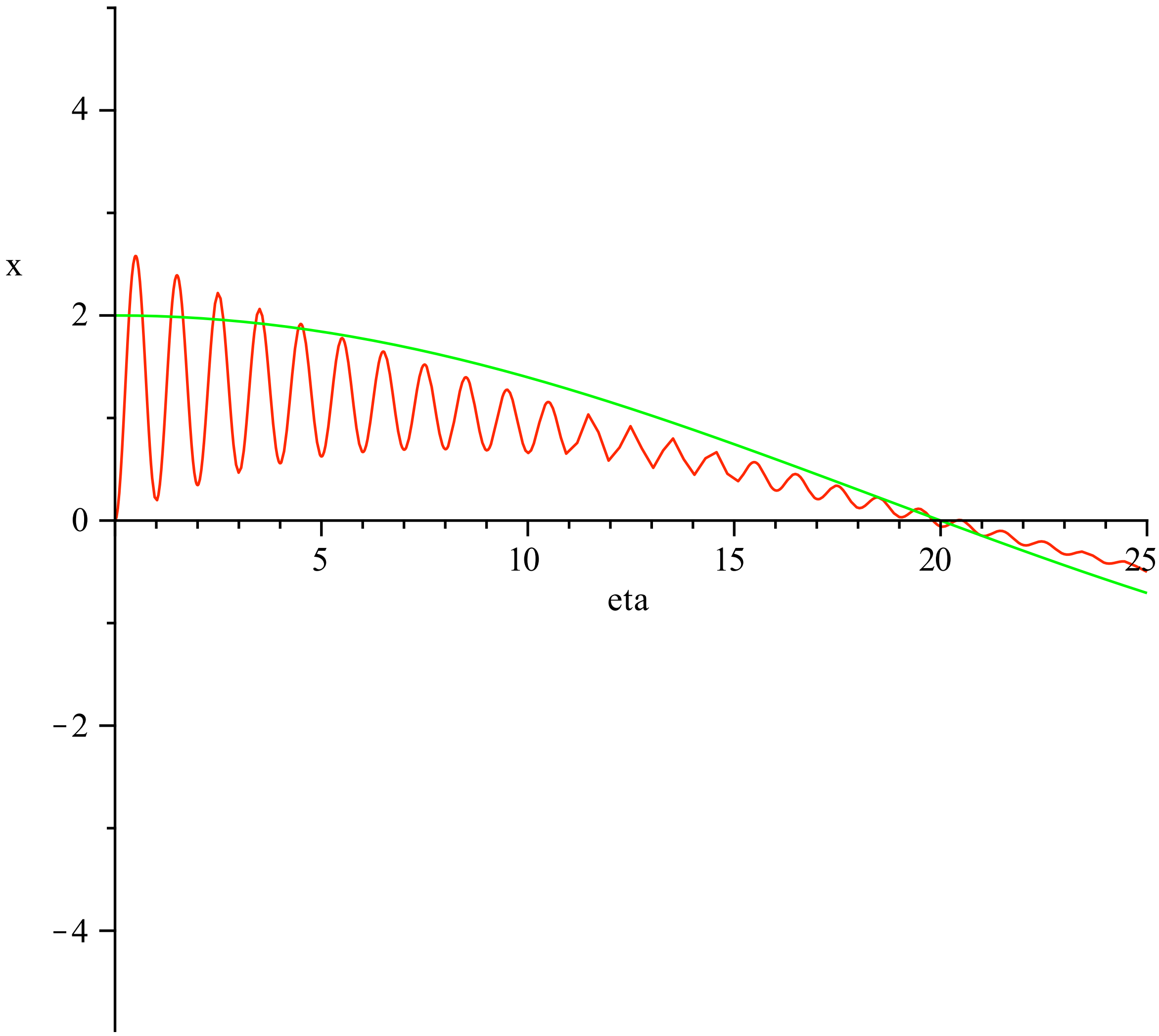
```
> solyfcn:=unapply(soly,omgT):
```

It is also convenient to define a function for the driving force,

```
> drivingfrc := omgT-> cos(omgT*eta)+cos(1.5*omgT*eta):
```

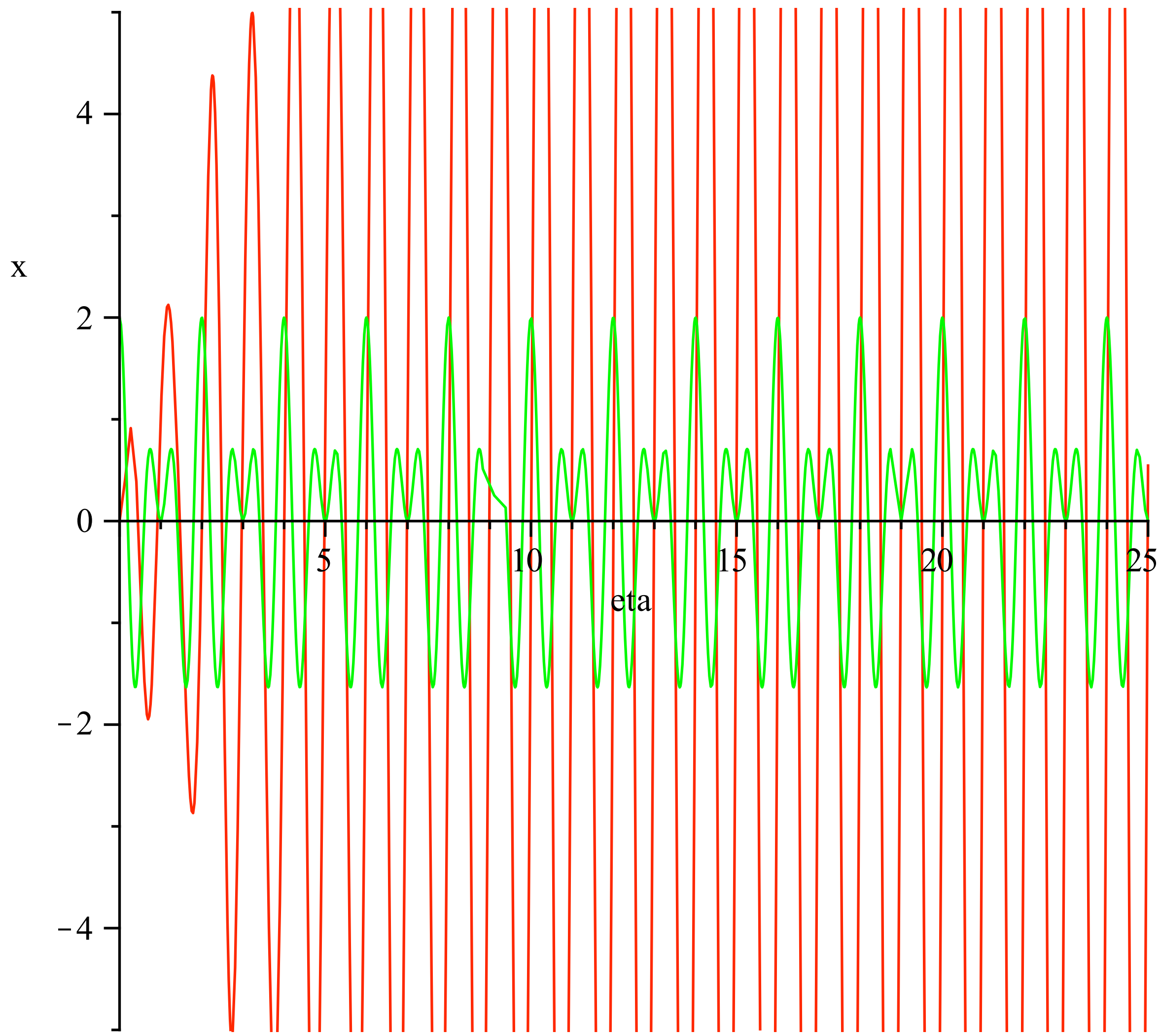
The plot below shows a low driving frequency, $\omega_D = 0.01\omega_N$. The red curve is the oscillator, the green curve is the driving force.

```
> plot([solyfcn(0.01*2*Pi),drivingfrc(0.01*2*Pi)],eta=0..25,x=-5.
  .5);
```



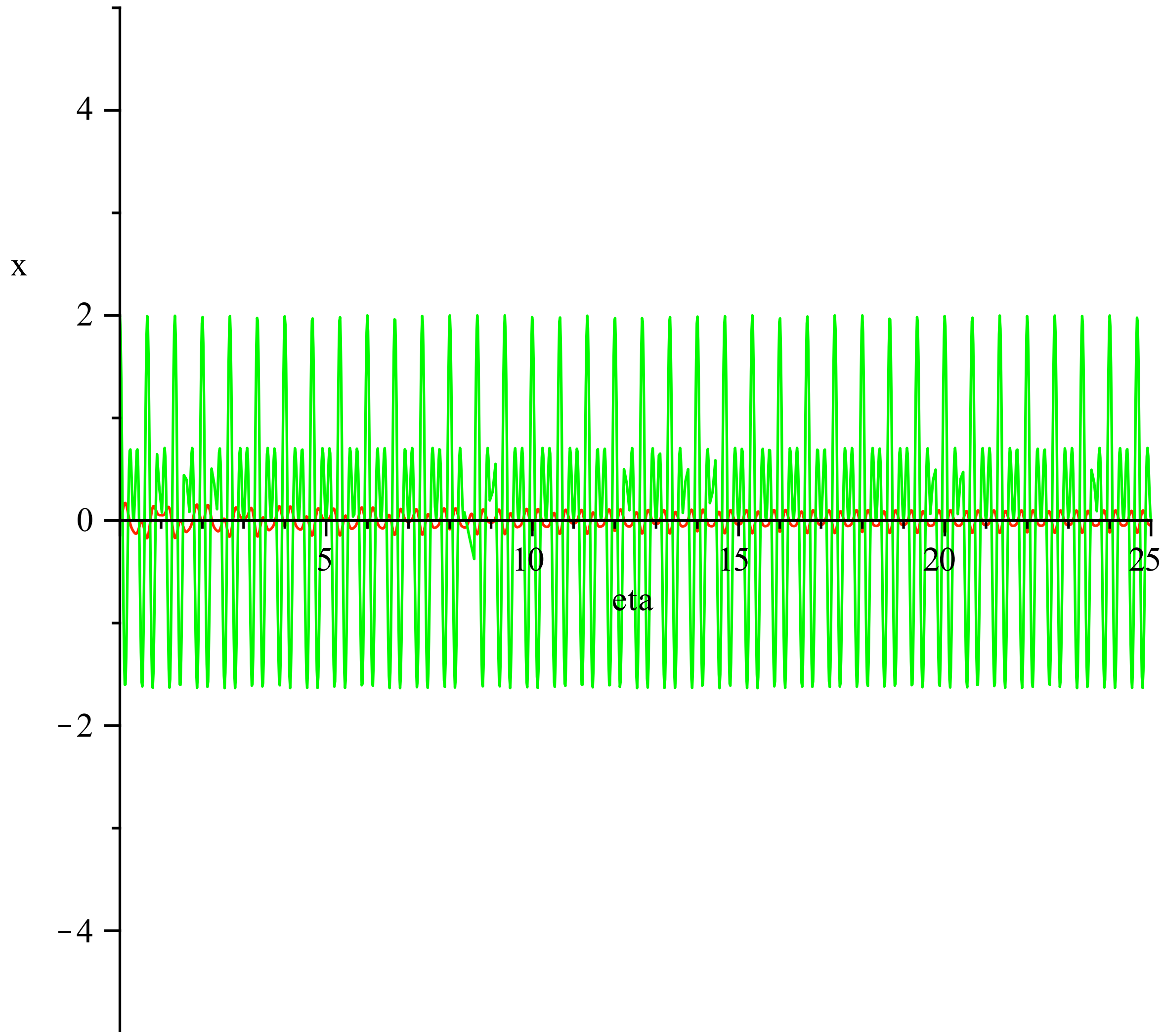
A similar plot, but now driving at the natural frequency: Note that the driving force is much more complicated looking than when there was only one codine driving term.

```
> plot([solyfcn(2*Pi),drivingfrc(2*Pi)],eta=0..25,x=-5..5);
```



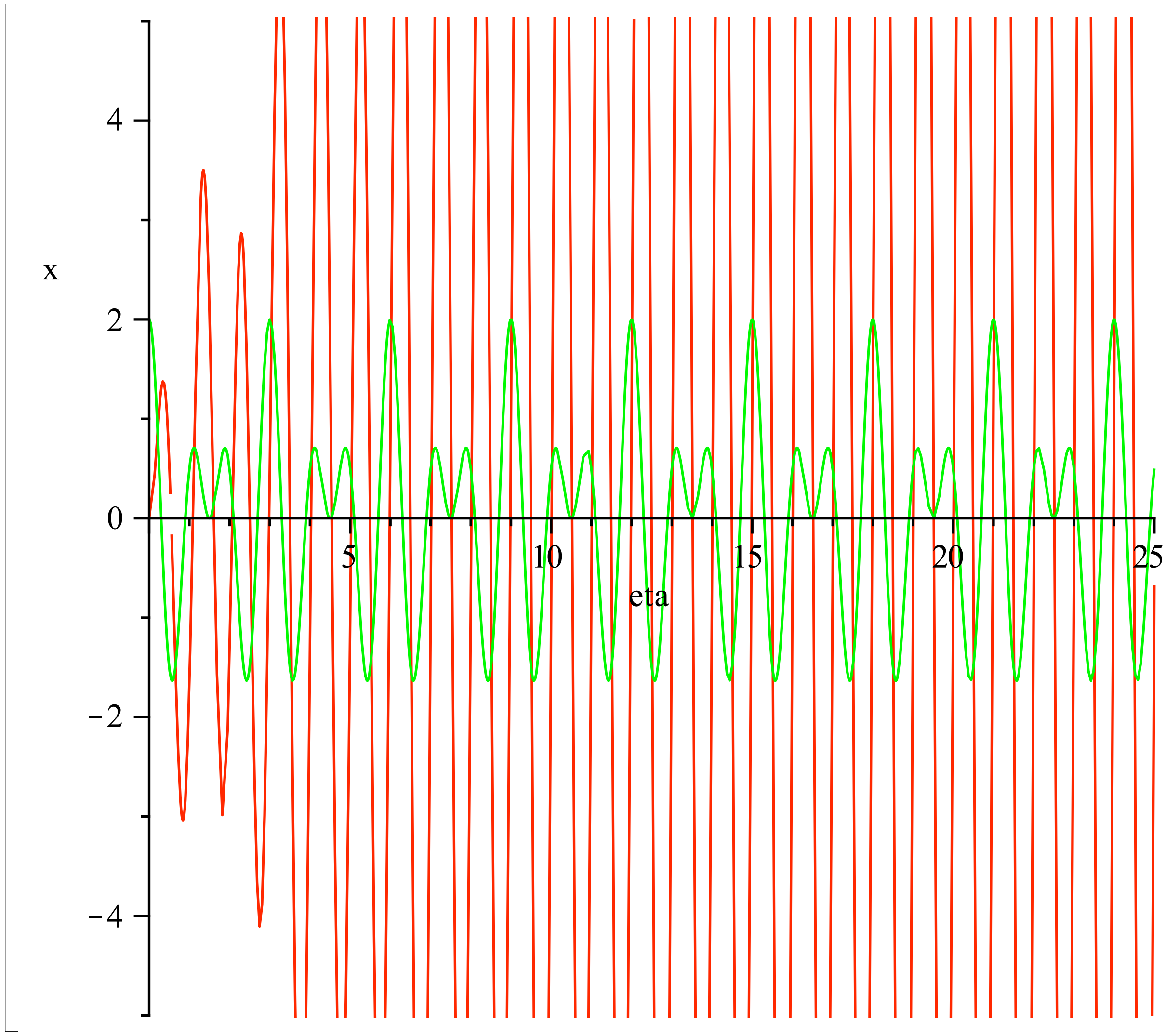
Finally, a graph of the response well above the natural frequency,
$\omega_D = 5\omega_N$. Again, the response is very small.

```
> plot([solyfcn(3*2*Pi),drivingfrc(3*2*Pi)],eta=0..25,x=-5..5);
```



It is interesting to note (but goes beyond what you had to do to do the problem) that there is a resonant response at two driving frequencies. The first occurs when $\omega_D = \omega_N$; . Then the first cosine driving term causes a resonant response. The second occurs when $1.5\ \omega_D = \omega_N$; then the second cosine term drives the oscillator at resonance. A plot of the solution at that frequency is below.

```
> plot([solyfcn(2*Pi/1.5),drivingfrc(2*Pi/1.5)],eta=0..25,x=-5..5);
```



x
4
2
0
−2
−4
5
10
15
20
25
eta

4. The sum of voltage drops in a circuit is zero, so in this problem

$$V_{\text{resist}} + V_{\text{cap}} + V_{\text{induct}} = 0. \tag{23}$$

Using the expressions for voltage given in the homework turns this into

$$IR + \frac{Q}{C} - L\frac{dI}{dt} = 0. \tag{24}$$

This doesn't look like the same equation as a harmonic oscillator; that equation has second derivatives in it. Taking the time derivative of (24) gives

$$R\frac{dI}{dt} + \frac{1}{C}\frac{dQ}{dt} - L\frac{d^2I}{dt^2} = -L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C} = 0. \tag{25}$$

In terms of the notation in the assignment, $B = -L$, $D = R$ and $E = 1/C$.

5. The modulus (or magnitude) of the complex number z is $|z|^2 = zz^*$.

   (a) The modulus of this number is $|z|^2 = (i/\sqrt{3})(-i/\sqrt{3}) = 1/3$. The number $i/\sqrt{3}$ is in rectangular form. The polar form is $z = re^{i\phi}$, where $r = |z| = \sqrt{1/3} = 1/\sqrt{3}$ and $\tan\phi = \text{Im}(z)/\text{Re(z)} = \infty$ so that $\phi = \pi/2$ and $z = e^{i\pi/2}/\sqrt{3}$.

   (b) The modulus is

$$|z|^2 = zz^* = (4e^{i\pi/4})(4e^{-i\pi/4}) = 16. \tag{26}$$

   This number is in polar form; you can write it in rectangular form using Euler's formula, $e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, so that $z = 2\sqrt{2}(1+i)$.

   (c) Get started with this one by eliminating the $1-i$ from the denominator,

$$\frac{2+i}{1-i}\frac{1+i}{1+i} = \frac{1+3i}{2}. \tag{27}$$

   The magnitude r is

$$r = \sqrt{|z|^2} = \sqrt{zz^*} = \sqrt{\frac{1+3i}{2}\frac{1-3i}{2}} = \sqrt{\frac{10}{4}} = \sqrt{\frac{5}{2}}, \tag{28}$$

and the polar angle is given by

$$\tan\phi = \mathrm{Im}(z)/\mathrm{Re(z)} = \frac{(3/2)}{(1/2)} = 3 \qquad \Longrightarrow \qquad \phi = \tan^{-1} 3 = 1.25, \tag{29}$$

so that

$$z = \sqrt{\frac{5}{2}} e^{1.25i}. \tag{30}$$