1. The Taylor series for cosine about $x_0 = 0$ is

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots . \quad (1)$$

Since the series contains only even powers of x we must use the second form given in the assignemnt of the esitmate for the error in the Taylor series,

$$|R_n(x)| = \frac{|f^{(n+2)}(x_0)|}{(n+2)!}|(x - x_0)^{(n+2)}|. \quad (2)$$

Rearranging to solve for $(x - x_0)$ gives

$$|x - x_0| = \left(\frac{(n+2)!}{|f^{(n+2)}(x_0)|}|R_n|\right)^{\frac{1}{n+2}} . \quad (3)$$

(a) For $n = 4$ and an error of $R_4 = 0.1$ we have

$$x = (6!(0.1))^{1/6} = 2.04, \quad (4)$$

which is close to the value we estimated in the previous problem set.

(b) For $n = 10$ the formula is

$$x = \left(\frac{12!}{|\cos^{(12)}(0)|}R_{10}\right)^{1/12} = 4.37, \quad (5)$$

where we have used an error of $R_{10} = 0.1$. Note that x is larger because n is larger and our series is more accurate. The Maple worksheet below explains the rest of this problem.

```
> restart;
```

Problem 1

Part b

We now calculate the actual position at which cosine and its tenth order approximation differ by 0.1.

```
> TenthOrder := convert(taylor(cos(x), x = 0, 11), polynom);
```

$$TenthOrder := 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10} \quad \textbf{(1.1.1)}$$

```
> k := unapply(TenthOrder, x);
```

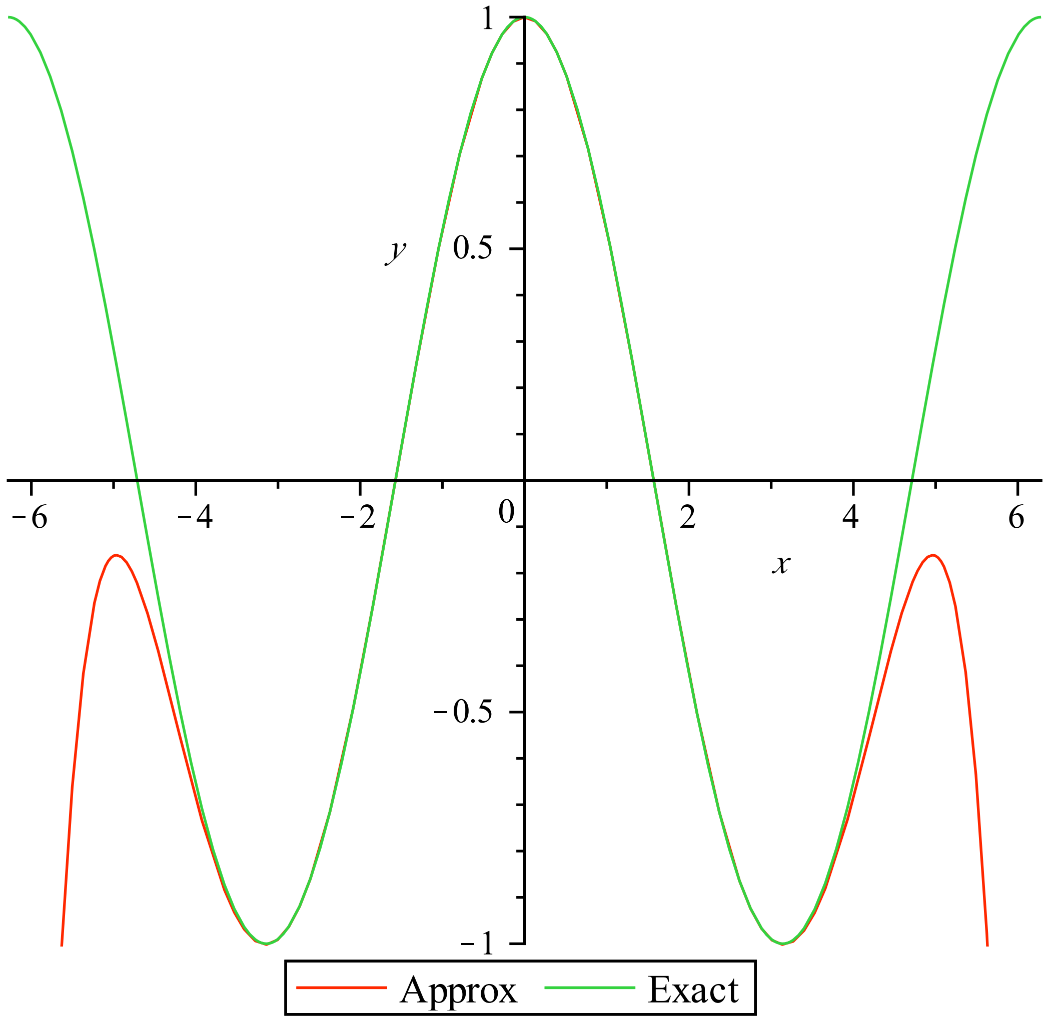
$$k := x \rightarrow 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10} \quad \textbf{(1.1.2)}$$

```
> fsolve(k(x)-cos(x) = 0.1, x);
```

Horror of horrors! Maple has failed us! Instead of solving the equation it gave back what we typed in. Why? Let's make a graph of the approximation and the cosine:

$$fsolve\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10} - \cos(x) = 0.1, x\right) \quad \textbf{(1.1.3)}$$

```
> plot([k(x), cos(x)], x =-2·π ..2·π, y =-1 ..1, legend = ["Approx",
      "Exact"]);
```



This explains the problem. Since our approximation is always less than cosine, $k(x) - \cos(x)$ is always negative, so it can never equal +0.1. How do we fix this up? The obvious fix is to replace 0.1 with -0.1. Instead, let's try something a bit more general. We will set $|k(x) - \cos(x)| = 0.1$ and since the absolute value is always positive, this will have a solution. Here we go:

```
> fsolve(abs(k(x)-cos(x))=0.1, x);
```

$$fsolve\left(\left|-1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{720}x^6 - \frac{1}{40320}x^8 + \frac{1}{3628800}x^{10} + \cos(x)\right| = 0.1, x\right) \quad \textbf{(1.1.4)}$$

Alas! Maple has failed us again! What do we try now? No, not the obvious fix, that would work, but do you really want to have to make a graph before you check your approximations? And besides, shouldn't Maple understand what we mean?

As our next step we will use solve instead of fsolve. The latter is a more restricted version of the former, so let's see what happens:

```
> sols := solve(abs(k(x)-cos(x))=0.1, x);
```

$$sols := 4.403304799 - 0.\,I, 4.243582323 + 1.156574224\,I \quad \textbf{(1.1.5)}$$

At last, a solution! Actually two....one is in the complex plane. The other seems strange; Maple reports it to be 4.403304799-0.I. Seems a lot like a real number, doesn't it? But Maple doesn't think so:

```
> whattype(sols[2]);
```

$$complex(extended\_numeric) \qquad (1.1.6)$$

Fortunately, we are more clever than the computer and recognize that the answer is

```
> ℜ(sols[2]);
```

$$4.243582323 \qquad (1.1.7)$$

The moral of the story? Maple is nice, Maple is convenient, but Maple is Not Smart and is no substitute for the most important part of any computer calculation: the part between the keyboard and the back of the chair.

You can actually force Maple to recognize what is obvious to us by using the simplify command:

```
> simplify(sols[2]);
```

$$4.243582323 + 1.156574224\,I \qquad (1.1.8)$$

In any event, if you are like me, by now you've forgotten what the point of this problem was. We were to compare the exact position at which the Taylor series and the cosine differed by 0.1, given in equation (2.2.11), with the value we found using our approximation, given in (2.2.4). Clearly our estimated position was pretty good!

Problem 2

Making the substitutions suggested in the problem is easy enough; dividing both sides of $V = V_0\left(1 - e^{-\frac{t}{RC}}\right)$ by V_0 gives $\frac{V}{V_0} = 1 - e^{-\frac{t}{RC}}$. Substituting in $v = \frac{V}{V_0}$ and $u = \frac{t}{RC}$ gives

```
> v := u→ 1-exp(-u);
```

$$v := u \rightarrow 1 - e^{-u} \qquad (2.1)$$

The first order approximation for the potential is, in terms of t and v,

```
> v1 := u→ u;
```

$$v1 := u \rightarrow u \qquad (2.2)$$

To calculate the range of t over which this is valid we need to calculate the second derivative of v, for which we don't even need Maple, since it is clearly $v'' = -e^{-u}$. The estimate for the value of u at which the approximation becomes off by more than 0.05 is then

```
> ubad := BigX(2, 0.05,-exp(0));
```

$$ubad := BigX(2, 0.05, -1) \qquad (2.3)$$

Expressed in terms of t this is $t = 0.316\ RC$. Now we use Maple to find the point at which the two differ by 0.05:

```
> fsolve(u-(1-exp(-u))=0.05, u);
```

$$0.3338105448 \qquad (2.4)$$

Once again, not bad!

Problem 3

Once again we are to check the range of validity of an approximation. In this case we want to know the speed at which the relativistic and non relativistic expressions for energy differ by more than 5% of the rest mass. Recall that the relativistic energy is given by

$$E = \frac{mc^2}{\sqrt{\left(1 - \left(\frac{v}{c}\right)^2\right)}}$$

and the to the lowest non-zero order in v the energy is approximately

$E_{approx} = mc^2 + \frac{mv^2}{2} = mc^2\left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right)$. I've rewritten the energy in the second form to make it easier to do the change of variables suggested in the problem. Setting $\mathcal{E} = \frac{E}{mc^2}$ and $\beta = \frac{v}{c}$ gives the expressions

```
> E_exact := β → (1-β^2)^(-1/2);
```

$$\mathcal{E}_{exact} := \beta \rightarrow \frac{1}{\sqrt{1 - \beta^2}} \quad \textbf{(3.1)}$$

and

```
> E_approx := β → 1 + 1/2 β^2;
```

$$\mathcal{E}_{approx} := \beta \rightarrow 1 + \frac{1}{2}\beta^2 \quad \textbf{(3.2)}$$

As before, we do this two ways, first with the approximate formula, and then by having Maple solve for the exact point at which the two energies differ by 0.05. Note that we need to be careful, as we were with the cosine. The Taylor series expansion of $\mathcal{E}_{exact}$ about zero is

```
> taylor(E_exact(β), β = 0, 5);
```

$$1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + O(\beta^6) \quad \textbf{(3.3)}$$

In the previous homework we kept the first two terms of (4.3), so we will use the β^4 term to estimate the error. This means calculating the fourth derivative of the exact energy. While we could do that by hand it is easier to just ask Maple (the unapply is there to allow us to use the fourth derivative as a function)

```
> E_fourth := unapply(diff(E_exact(β), β$4), β);
```

$$\mathcal{E}_{fourth} := \beta \rightarrow \frac{105\,\beta^4}{\left(1-\beta^2\right)^{9/2}} + \frac{90\,\beta^2}{\left(1-\beta^2\right)^{7/2}} + \frac{9}{\left(1-\beta^2\right)^{5/2}} \quad \textbf{(3.4)}$$

so that the value of β at which we expect our approximation to fail is given by

```
> β_bad := BigX(6, 0.05, E_fourth(0));
```

$$\beta_{bad} := BigX(6, 0.05, 9) \quad \textbf{(3.5)}$$

This is a problem.....remember that $\beta = \frac{v}{c}$ so that apparently our approximation breaks down for particles that move 1.26 times the speead of light. Let's try having maple calculate the value for us

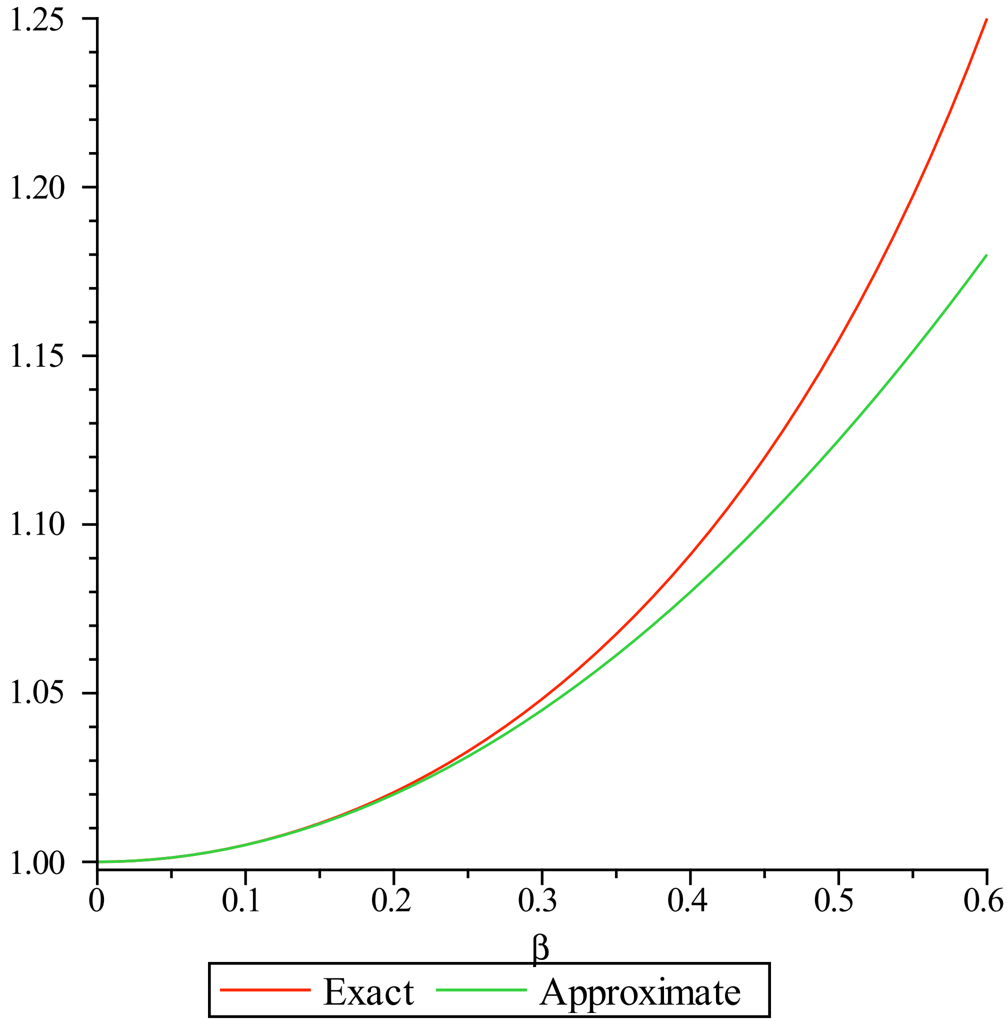
```
> fsolve(E_exact(β) - E_approx(β) = 0.05, β);
```

(3.6)

$$0.5595014203 \tag{3.6}$$

This seems a bit surprising. It suggests that up to speeds of roughly half the speed of light the only change we need to make to the classical formula for the kinetic energy is to add the particle's rest mass energy mc^2, so let's make a plot to be sure:

> plot([$\mathcal{E}_{exact}(\beta)$, $\mathcal{E}_{approx}(\beta)$], β = 0..0.6, legend = ["Exact", "Approximate"]);



A couple of comments are in order.
First, our approximate formula for the point at which a series approximation breaks down doesn't always work. In fact, there is a remainder theorem that says that the actual remainder in a Taylor series of order n is

$$R_n(x) = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!} \left|(x-x_0)^{n+1}\right|$$

where c is some number between x_0 and x. In the homework assignment, to simplify matters, we set $c = x_0$ and in the couple of problems above that worked well. Here it does not.

Second, relativistic effects become important at around $\beta = 0.05$ or maybe $\beta = 0.1$, a much smaller number than we found. Not surprising, since a 5% error is fairly large.

Problem 4

A complex problem, for a change of pace. We are told that the electric field in an electromagnetic wave can be written, in some circumstances, as

$E = E_0 e^{i\omega t}$. We will begin by defining a Maple function for this.

```
> Efield := t→E0exp(I · ω ·t);
```

$$Efield := t \to E_0\, e^{I\,\omega\, t} \tag{4.1}$$

We plot this for a few values of t from zero to one period,

```
> complexplot([Efield(0), Efield(0.3·2·π/ω), Efield(.5·2·π/ω)]);
```

$$complexplot\left(\left[E_0, E_0\, e^{0.6\, I\,\pi}, E_0\, e^{1.0\, I\,\pi}\right]\right) \tag{4.2}$$

Alas, no plot. Why? Because Maple does not know how to plot E_0 so we must either make it assume a value, or plot the field divided by E_0,

```
> complexplot([Efield(0)/E0, Efield(0.3·2·π/ω)/E0, Efield(.5·2·π/ω)/E0]);
```

$$complexplot\left(\left[1, e^{0.6\, I\,\pi}, e^{1.0\, I\,\pi}\right]\right) \tag{4.3}$$

Aaargghhh! Still no plot! Apparently we must force Maple to evaluate these expressions as complex numbers....

```
> complexplot([evalc(Efield(0)/E0), evalc(Efield(0.3·2·π/ω)/E0),
    evalc(Efield(.5·2·π/ω)/E0)], -2 ..2, style = point);
```

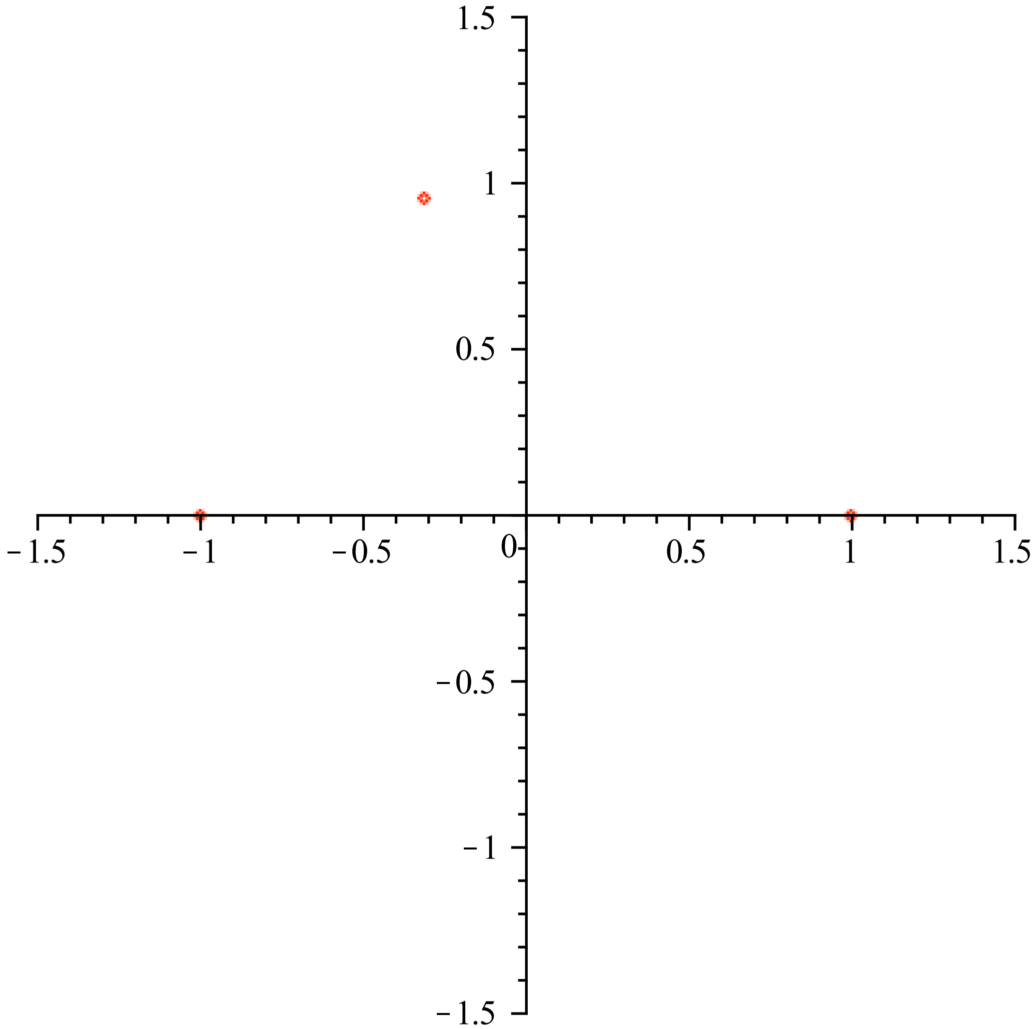
$$complexplot\left(\left[1, \cos(0.6\,\pi) + I\sin(0.6\,\pi), -1\right], -2..2, style = point\right) \tag{4.4}$$

```
> complexplot([1, cos(0.6 π) + I sin(0.6 π), -1], -2 ..2, style
    = point);
```

$$complexplot\left(\left[1, \cos(0.6\,\pi) + I\sin(0.6\,\pi), -1\right], -2..2, style = point\right) \tag{4.5}$$

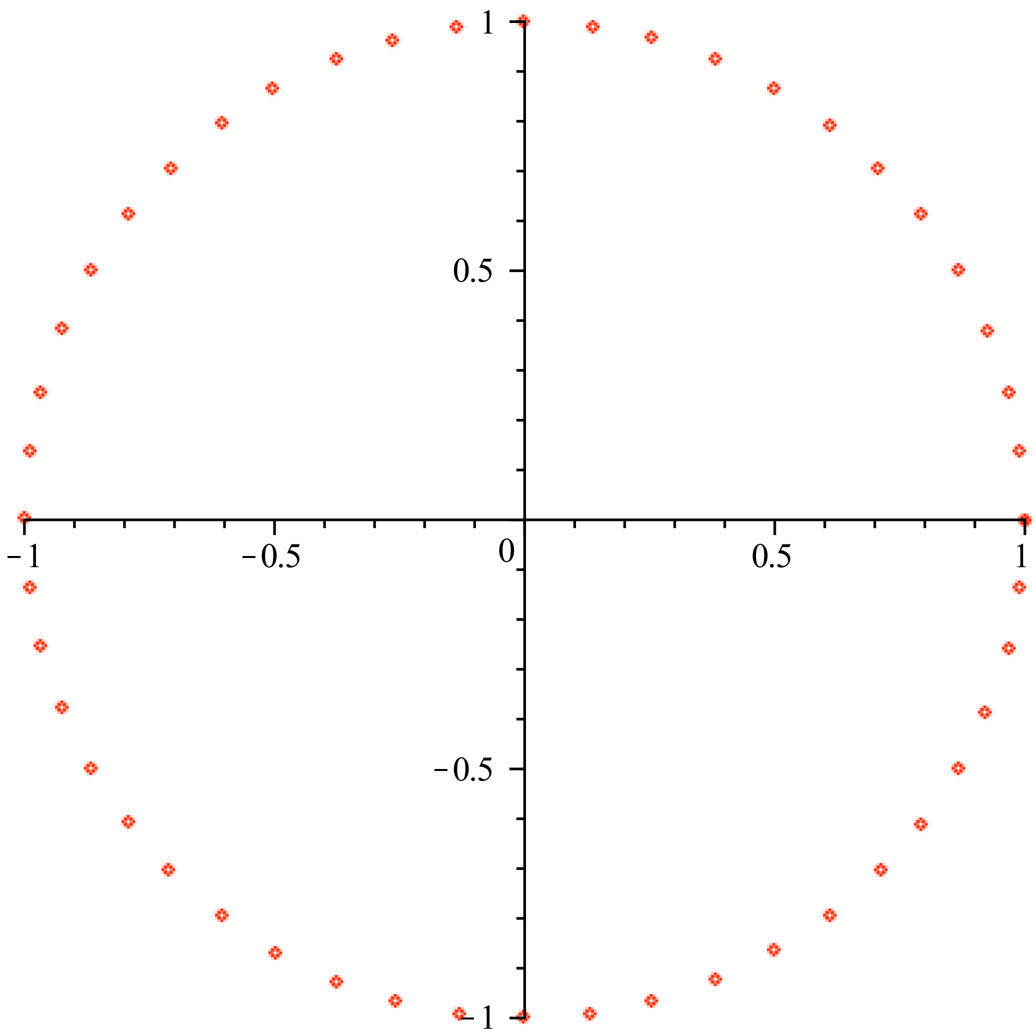
Aha! Having read the help for complexplot I now know I need to call with(plots) first:

```
> with(plots) :
> complexplot([Efield(0)/E0, Efield(0.3·2·π/ω)/E0, Efield(.5·2·π/ω)/E0], -1.5
    ..1.5, -1.5 ..1.5, style = point);
```



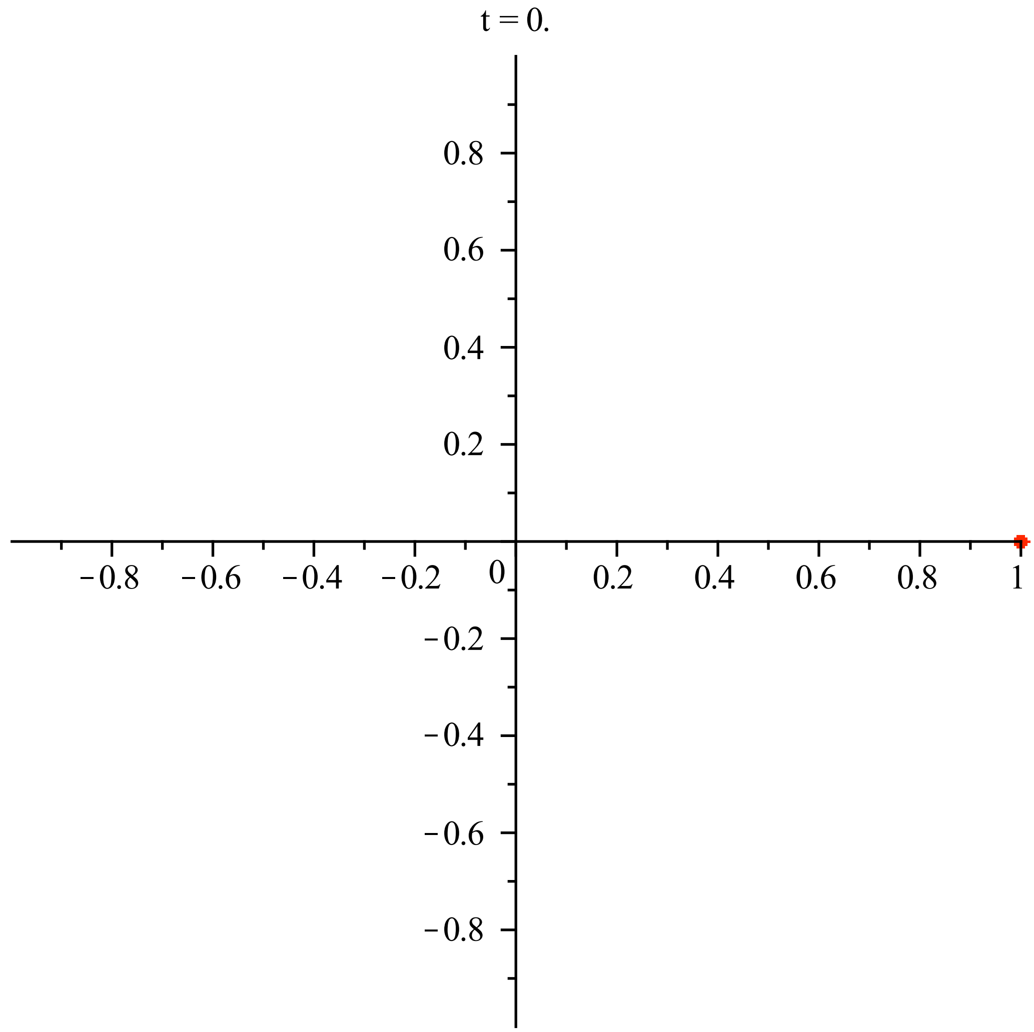
I don't know about you, but the shape is not obvious to me. Let's try this again, this time with a parameter for the fraction of a period that we want.

> **complexplot** $\left(\dfrac{\mathbf{Efield}\left(\dfrac{\mathbf{f}\cdot \mathbf{2}\cdot\pi}{\omega}\right)}{\mathbf{E_0}}, \mathbf{f = 0 .. 1, style = point}\right)$;



Apparently the shape is a circle, which ought not to be too surprising, since the magnitude of $\frac{E}{E_0}$ is one. We can also animate this with the following command. Note that you must click on the plot and then controls for the movie will appear in the tool bar at the top.

$$> \mathbf{animate}\left(\mathbf{complexplot}, \left[\frac{\mathbf{Efield}\left(\frac{\mathbf{t}\cdot 2\cdot\pi}{\omega}\right)}{\mathbf{E_0}}, \mathbf{x}=-1..1, \mathbf{style}=\mathbf{point}\right], \mathbf{t}=0..1, \mathbf{frames}=50\right);$$


t = 0.
0.8
0.6
0.4
0.2
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
−0.2
−0.4
−0.6
−0.8

>

5. We are to find solutions to the differential equation

$$2\frac{d^5x}{dt^5} - 5\frac{d^4x}{dt^4} + 2\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - x = 0, \tag{6}$$

which has solutions of the form $x = e^{at}$. Putting this solution into the differential equation gives

$$2a^5x - 5a^4x + 2a^3x + 2a^2x - x = 0 \Rightarrow 2a^5 - 5a^4 + 2a^3 + 2a^2 - 1 = 0. \tag{7}$$

The solutions of this are, according to Maple,

$$1,\ 1,\ 1/6\sqrt[3]{73+6\sqrt{87}} + \frac{13}{6}\frac{1}{\sqrt[3]{73+6\sqrt{87}}} + 1/6 \tag{8}$$

$$-1/12\sqrt[3]{73+6\sqrt{87}} - \frac{13}{12}\frac{1}{\sqrt[3]{73+6\sqrt{87}}} + 1/6 + 1/2\,i\sqrt{3}\left(1/6\sqrt[3]{73+6\sqrt{87}} - \frac{13}{6}\frac{1}{\sqrt[3]{73+6\sqrt{87}}}\right), \tag{9}$$

$$-1/12\sqrt[3]{73+6\sqrt{87}} - \frac{13}{12}\frac{1}{\sqrt[3]{73+6\sqrt{87}}} + 1/6 - 1/2\,i\sqrt{3}\left(1/6\sqrt[3]{73+6\sqrt{87}} - \frac{13}{6}\frac{1}{\sqrt[3]{73+6\sqrt{87}}}\right). \tag{10}$$

The last two of these solutions are complex, and since we know that exponentials of the form $e^{i\theta}$ can be written as sines and cosines, the two complex solutions are oscillatory.