Physics 350 Problem Set 3 (Spring Semester 2009)
Due Thu., February 5 at 4:30PM

In the first few problems we will explore the accuracy of Taylor series expansions as a follow up to what you did in lab this week. The actual problems follow the bit of background below.

Suppose you have calculated the Taylor series expansion of the function $f(x)$ to $n$th order. Then we can write $f(x)$ as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x), \quad (1)$$

where the remainder $R_n(x)$ is sum of the terms we haven't calculated. In other words, $R_n(x)$ is the error we are making in calculating $f$ using the first $n$ terms of the expansion instead of the exact function. One *estimate* of the magnitude of $R_n$ is

$$|R_n(x)| \approx \frac{|f^{(n+1)}(x_0)|}{(n+1)!}|(x - x_0)^{n+1}|; \quad (2)$$

this estimate makes sense because it is the biggest of the terms we are leaving out of the series.

Suppose the biggest error you can tolerate in the approximating a function as a Taylor series is $\delta$ (in other words, $|R_n| = \delta$). If we rearrange (2) to solve for the value of $|x - x_0|$ you get

$$|x - x_0| = \left( \frac{(n_+1)!\delta}{f^{(n+1)}(x_0)} \right)^{1/(n+1)}, \quad (3)$$

giving you an estimate of the largest $x - x_0$ can be without making the error too large.

In some cases the formula above does not work. For example, the Taylor series expansion of $\sin x$ and $x_0 = 0$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \quad (4)$$

If you approximate $\sin x$ by ending the series at 5th order, the error you get is not given by the expression (2) for$R_5$, because that formula would give

$$|R_5(x)| = \frac{|f^{(6)}(x_0)|}{6!}|(x - x_0)^6| = 0, \quad (5)$$

since $f^{(6)} = -\sin 0 = 0$. Here the biggest term we are leaving out is the $7^{\text{th}}$ order term, so the error in this specific case of ending the series at $5^{\text{th}}$ is

$$R_5(x) \approx \frac{|f^{(7)}(x_0)|}{7!}|(x - x_0)^7|. \tag{6}$$

More generally, for a series like this that has only odd terms or only even terms, the correct starting point for estimating the error is

$$|R_n(x)| = \frac{|f^{(n+2)}(x_0)|}{(n+2)!}|(x - x_0)^{(n+2)}| = 0, \tag{7}$$

1. (a) Consider $\cos x$ expanded about $x_0 = 0$ to fourth order. Use the preceeding discussion as a guide to calculate the largest that $x$ can be so that the error is no larger than 0.1. Explicitly state whether you are basing your estime of $x$ on (2) or on (7).

   (b) What is the largest $x$ if expanded to tenth order? Now use Maple to calculate the position at which the error is 0.1 like you did in lab. Compare the two values–the one from Maple and the one from the formula–for $x$ and comment.

2. In the last problem set we considered the voltage drop across a capacitor in an $RC$ circuit, given by $V = V_0(1 - e^{-t/(RC)})$, where $V_0$ is the final voltage drop across the capacitor. You found last week that the Taylor series expansion of this to first order is $V \approx V_0 t/(RC)$. Here we want to get some sense of how accurate this approximation is.

   - It will be easiest to work with the exact formula for the potential if we rewrite it in terms of new variables. Make the substitutions $u = t/(RC)$ and $v = V/V_0$, and write the expression for voltage in terms of the new variables $v$ and $u$.
   - Use the preceeding discussion as a guide to calculate the largest time at which the first order Taylor series approximation for $v$ is accurate to within an error of $\delta = .05$. Explicitly state whether you are basing your estime of $t$ on (2) or on (7). Check your result by using Maple to find the point at which the Taylor series approximation differs from by .05.

3. Figure out the speed at which the approximate expression for the relativistic energy of a particle that we obtained in the first homework

differs from the exact expression by $0.05mc^2$. Like in the previous problem you will need to introduce new, dimensionless, variables. For this problem make the substitution $\beta = v/c$ and $\mathcal{E} = E/(mc^2)$.

4. Now a complex problem...the electric field in a an electromagnetic wave is the real part of $E = E_0 e^{i\omega t}$, where $\omega$ is the frequency of the wave.

   - Plot the position of the electric field in the complex plane for a few values between $t = 0$ and $t = \tau$, where $\tau = 2\pi/\omega$ is the period of the wave. What shape do these points lie on?
   - You can make a movie showing the motion of the electric field over time. Use the `animate` command in Maple; to use this command you need to have the command `with(plots):` in the first couple of lines of your worksheet *before* you use `animate`.

5. Consider the differential equation, similar to the one we considered in class,

$$2\frac{d^5x}{dt^5} - 5\frac{d^4x}{dt^4} + 2\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - x = 0. \tag{8}$$

   In class we considered solutions of the form $x = e^{at}$.

   (a) (Do this by hand!) Derive the algebraic equation that determines $a$.

   (b) Solve that equation for $a$ using Maple.

   (c) How many of the solutions $x$ are oscillatory (that is, contain a sine or cosine)?