Solutions written by Matthew Craig and Juan Cabanela

Problem 1

After initiallizing **Maple** with the following

```
> restart;
> with(plots):
```

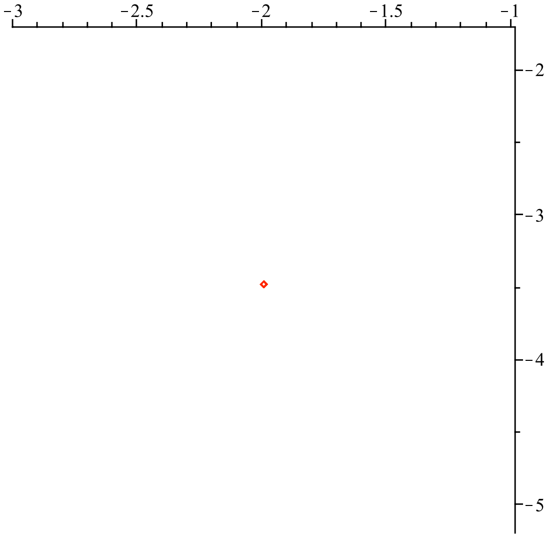
For the first complex number, I defined

```
> first := 4*(cos(2*Pi/3)-I*sin(2*Pi/3));
```

$$\mathit{first} := -2 - 2\,i\sqrt{3}$$

And used the **complexplot** command as follows:

```
> complexplot(first,x=-3..3,style=point);
```

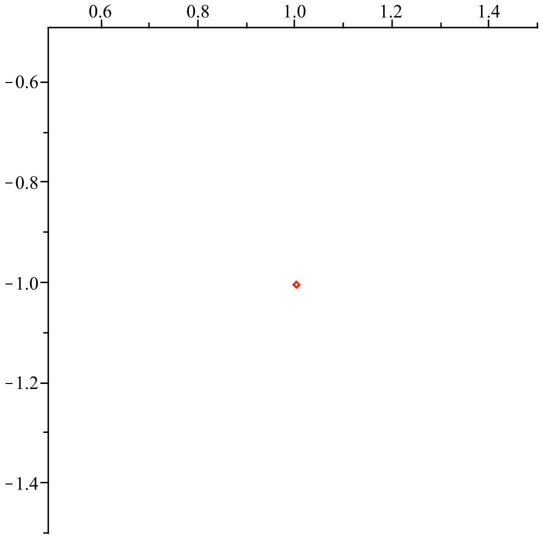


For the other three complex numbers, I also created definitions and plotted them:

```
> second := sqrt(2)*exp(-I*Pi/4);
```

$$\mathit{second} := \sqrt{2}\left(1/2\,\sqrt{2} - 1/2\,i\sqrt{2}\right)$$

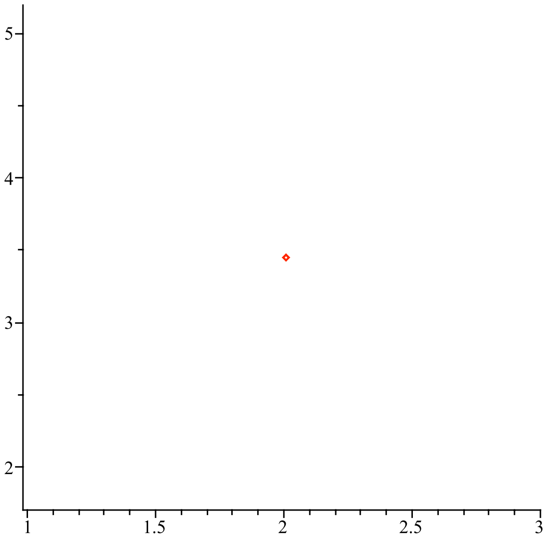
```
> complexplot(second,x=-3..3,style=point);
```



```
>  third := (I+sqrt(3))^2;
```

$$third := \left(i+\sqrt{3}\right)^2$$

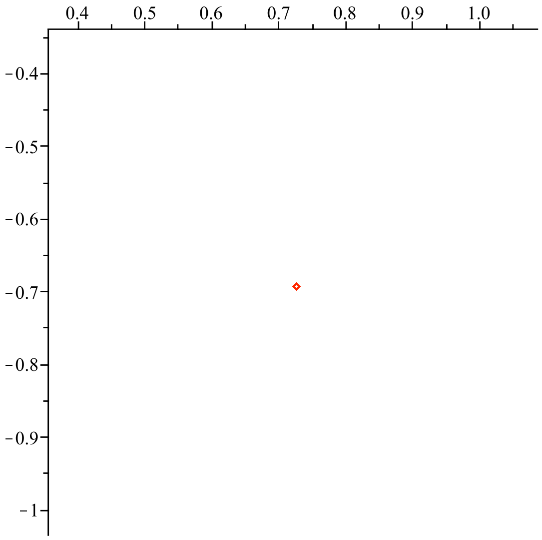
```
>  complexplot(third,x=-3..3,style=point);
```



```
>  fourth := (5-2*I)/(5+2*I);
```

$$fourth := \frac{21}{29} - \frac{20}{29}\,i$$

```
>  complexplot(fourth,x=-3..3,style=point);
```



Problem 2

We are to find all solutions to the equation $(x+iy)^3 = -1$. Following the hint in the problem we will first find solutions to $z^3 = -1$, where z is the solution in polar form. The polar form of -1 is

$$-1 = e^{i(\pi+2\pi n)}, \tag{1}$$

where $n = 0, 1, 2, \ldots$ is an integer, so the solutions z are given by

$$z = (-1)^{1/3} = e^{i(\pi+2\pi n)/3} = e^{i\pi(1+2n)/3}. \tag{2}$$

The three distinct solutions to this equation are given by $n = 0, 1$ and 2, so that

$$z = e^{i\pi/3}, \quad e^{i\pi}, \quad e^{5i\pi/3}. \tag{3}$$

Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, we can obtain the real and imaginary part of each solution:

$$e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad\Rightarrow\quad x = \frac{1}{2} \text{ and } y = \frac{\sqrt{3}}{2} \tag{4}$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 \quad\Rightarrow\quad x = 1 \text{ and } y = 0 \tag{5}$$

$$e^{5i\pi/3} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i \quad\Rightarrow\quad x = \frac{1}{2} \text{ and } y = -\frac{\sqrt{3}}{2} \tag{6}$$

Problem 3

a) This problem is easier if we write the numerator and denominator in polar form using Euler's formula,

$$\frac{1+i}{1-i} = \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{i\pi/2} = i, \tag{7}$$

so that

$$\left(\frac{1+i}{1-i}\right)^4 = i^4 = 1. \tag{8}$$

b) This problem is alos easier if we begin by writing the complex number in polar form,

$$\frac{i\sqrt{2}}{1+i} = \frac{\sqrt{2}e^{i\pi/2}}{\sqrt{2}e^{i\pi/4}} = e^{i\pi/4}, \tag{9}$$

so that

$$\left(\frac{i\sqrt{2}}{1+i}\right)^{12} = (e^{i\pi/4})^{12} = e^{3i\pi} = e^{i\pi} = -1. \tag{10}$$

Problem 4

The voltage drop across an initially uncharged capacitor in an RC circuit is given by

$$V(t) = V_0\left(1 - e^{-t/RC}\right); \tag{11}$$

we are to do the Taylor series expansion of this about $t = 0$. Recall that the Tylor series of a function $f(x)$ about the point x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(2)}(x_0)}{n!}(x - x_0)^n, \tag{12}$$

where $f^{(n)}$ is the n^{th} derivative of f. The first few terms of the expansion of V are

$$\begin{array}{ccc}
\text{Derivative} & \text{evaluated at } t=0 & \text{Term in series} \\
V^{(0)}(t) = V_0(1-e^{-t/RC}) & V^{(0}(0) = 0 & 0 \\
V^{(1)}(t) = \frac{V_0}{RC}e^{-t/RC} & V^{(1)}(0) = \frac{V_0}{RC} & \frac{V_0}{RC} \\
V^{(2)}(t) = -\frac{V_0}{(RC)^2}e^{-t/RC} & V^{(2)}(0) = -\frac{V_0}{(RC)^2} & -\frac{1}{2}\frac{V_0}{(RC)^2},
\end{array} \tag{13}$$

so that

$$V(t) \approx \frac{V_0}{RC}t - \frac{1}{2}\frac{V_0}{(RC)^2}t^2. \tag{14}$$

Problem 5

Part a

```
>   f := x-> cos(x);
```

$$f := x \mapsto \cos(x)$$

```
>   f2 :=convert(taylor(f(x),x=0,3),polynom);
```

$$f2 := 1 - 1/2\,x^2$$

```
>   f4 := convert(taylor(f(x),x=0,5),polynom);
```

$$f4 := 1 - 1/2\,x^2 + 1/24\,x^4$$

```
>   f6 := convert(taylor(f(x),x=0,7),polynom);
```

$$f6 := 1 - 1/2\,x^2 + 1/24\,x^4 - \frac{1}{720}\,x^6$$

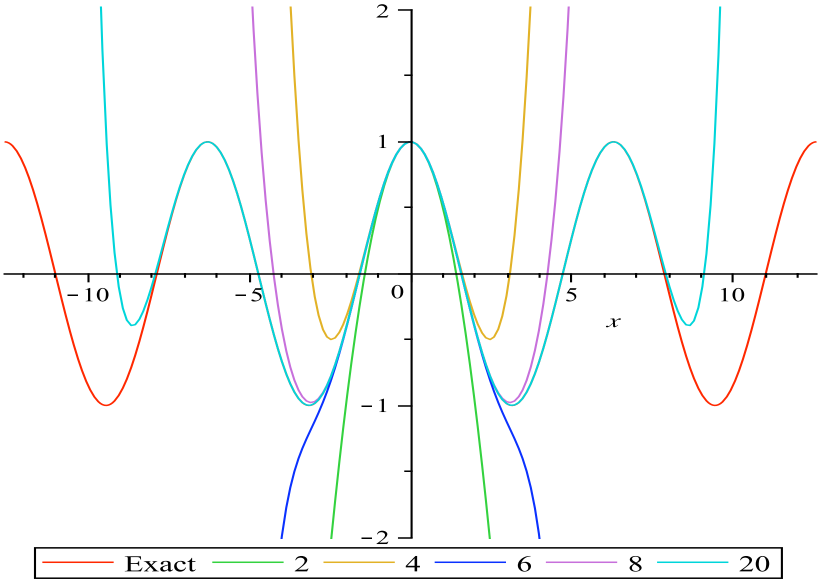
```
>   f8 := convert(taylor(f(x),x=0,9),polynom);
```

$$f8 := 1 - 1/2\,x^2 + 1/24\,x^4 - \frac{1}{720}\,x^6 + \frac{1}{40320}\,x^8$$

```
>   f20 := convert(taylor(f(x),x=0,21),polynom);
```

$$
\begin{aligned}
f20 \;:=\; & 1-1/2x^2+1/24x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8\\
& -\frac{1}{3628800}x^{10}+\frac{1}{479001600}x^{12}-\frac{1}{87178291200}x^{14}\\
& +\frac{1}{20922789888000}x^{16}-\frac{1}{6402373705728000}x^{18}\\
& +\frac{1}{2432902008176640000}x^{20}
\end{aligned}
$$

```
>  plot([f(x),f2,f4,f6,f8,f20],x=-4*Pi..4*Pi,-2..2,legend=["Exact","2","4","6",
```



Part b As the order of the approximation increases, the position at which the approximation differs from the exact function increases. The table below shows the approximate position at which the approximation differs from the exact curve. For the purposes of this problem I am judging whether the two curves differ by eye.
Second order diverges significantly at $x \approx 1.35$
Fourth order diverges significantly at $x \approx 1.9$
Sixth order diverges significantly at $x \approx 2.5$
Eighth order diverges significantly at $x \approx 3$
Twentieth order diverges significantly at $x \approx 8.2$