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Problem 1
We are to find the Taylor series expansion of the function $f(x) = x^{1/3}$ about the point $x_0 = 8$. Recall that the Taylor series expansion is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{1}$$

The first step is to calculate the derivatives of f through the third derivative. The derivatives are shown below,

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3}x^{-2/3}, \quad f''(x) = -\frac{2}{9}x^{-5/3}, \text{ and } f'''(x) = \frac{10}{27}x^{-8/3}. \tag{2}$$

Evaluating each of these at $x = 8$ gives $f(8) = 2$, $f'(8) = 1/12$, $f''(8) = -1/144$ and $f'''(8) = 5/3456$, so that the Taylor series is

$$f(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3 + \mathcal{O}((x-8)^4). \tag{3}$$

Problem 2
We are to find the Taylor series expansion of the exponential, $f(x) = e^x$ about the point $x_0 = 0$. The formula for the Taylor series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{4}$$

Recall that the first derivative is given by $f'(x) = e^x$; the derivative of the exponential is itself. Clearly the n^{th} derivative is also an exponential, so that $f^{(n)}(0) = 1$, and the Taylor series expansion is

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}x^n. \tag{5}$$

Problem 3
The relativistic energy of a particle is given by

$$E = \frac{m_0 c^2}{\sqrt{1 - (v/c)^2}}. \tag{6}$$

We are to Taylor expand this energy E about $v_0 = 0$, keeping terms through second order. The Taylor series expansion is

$$E = \sum_{n=0}^{\infty} \frac{E^{(n)}(v_0)}{n!}(v - v_0)^n, \tag{7}$$

where $E^{(n)}$ is the n^{th} derivative of E with respect to v.

The first derivative of E is

$$\frac{dE}{dv} = -\frac{1}{2}m_0c^2\left(1-(v/c)^2\right)^{-3/2}\left(-\frac{2v}{c^2}\right) = m_0v\left(1-(v/c)^2\right)^{-3/2}. \tag{8}$$

The second derivative of E is the derivative of (8) with respect to v, which is given by (do not forget to use the product rule in evaluating this derivative)

$$\frac{d^2E}{dv^2} = m_0\left(\left(1-(v/c)^2\right)^{-3/2} + v\left(-\frac{3}{2}\right)\left(1-(v/c)^2\right)^{-5/2}\left(-\frac{2v}{c^2}\right)\right) \tag{9a}$$

$$= m_0\left(1-(v/c)^2\right)^{-5/2}\left(1-(v/c)^2+3(v/c)^2\right) \tag{9b}$$

$$= m_0\left(1-(v/c)^2\right)^{-5/2}\left(1+2(v/c)^2\right). \tag{9c}$$

To evaluate the coefficients in the Taylor series we set $v = 0$ in the derivatives to get $E(0) = m_0c^2$, $E'(0) = 0$, and $E''(0) = m_0$. The Taylor series expansion is then

$$E = m_0c^2 + \frac{1}{2}m_0v^2 + \mathcal{O}(v^4); \tag{10}$$

the next non-zero order is v^4 not v^3 because it turns out the third derivative of E is proportional to v, and so is zero at $v = 0$. The expression for energy we have obtained is the rest energy of the particle plus its classical kinetic energy. Note that the problem is easier to do if we make the substitution $z = (v/c)^2$ at the beginning and expand in powers of z.

Problem 4

This problem is a relatively straightforward integration

```
> int(t*cos(t)*exp(-t),t);
```

$$-1/2\, t\cos(t)\,\mathrm{e}^{-t} - (-1/2\, t - 1/2)\,\mathrm{e}^{-t}\sin(t)$$

though you can simplify it a bit by factoring:

```
> factor( %);
```

$$-1/2\,\mathrm{e}^{-t}\left(t\cos(t)-\sin(t)\,t-\sin(t)\right)$$

Problem 5

The key to this problem is to set reasonable assumptions for all free parameters (m, k, and T),

```
> assume(m>0,k>0,T>0);
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Note that several constants cancel out in the integral,

$$\frac{\int 4\,\pi N\;v^4\;e^{-\frac{mv^2}{2\,k\,T}}\,dv}{\int 4\,\pi N\;v^2\;e^{-\frac{mv^2}{2\,k\,T}}\,dv}=\frac{4\,\pi N}{4\,\pi N}\frac{\int v^4\;e^{-\frac{mv^2}{2\,k\,T}}\,dv}{\int v^2\;e^{-\frac{mv^2}{2\,k\,T}}\,dv}=\frac{\int v^4\;e^{-\frac{mv^2}{2\,k\,T}}\,dv}{\int v^2\;e^{-\frac{mv^2}{2\,k\,T}}\,dv}.$$

Writing the integral is a bit easier if we define the function below

```
> g:= v^2 * exp(-m*v^2/(2*k*T));
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$$g\;:=\;v^2\mathrm{e}^{-1/2\,\frac{mv^2}{kT}}$$

which makes the integral

```
> int(v^2*g,v=0..infinity)/int(g,v=0..infinity);
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$$3\,\frac{kT}{m}$$

Problem 6

A figure showing each of the complex numbers in this problem in the complex plane is below.

a) This complex number is already in rectangular form, which makes it easy to find the real and imaginary parts. Writing $z=4\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$ we have $x=\Re(z)=4\cos(2\pi/3)=-2$ and $y=\Im(z)=-4\sin(2\pi/3)=-2\sqrt{3}$. The modulus is $r=\sqrt{x^2+y^2}=4$ and the phase angle is $\theta=\tan^{-1}\frac{-2\sqrt{3}}{-2}=\tan^{-1}\sqrt{3}=\frac{4\pi}{3}=240°$, so that $z=4e^{4i\pi/3}$. Note that the correct angle is *not* the answer your calculator probably gave; most likely it said $\tan^{-1}\sqrt{3}=\pi/3=60°$. That is not correct because the complex number we want is in the third quadrant because $x<0$ and $y<0$. In any event, the complex conjugate of z is $\bar{z}=4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

b) In this case the complex number is provided in polar form,

$$z=\sqrt{2}e^{-i\pi/4}=\sqrt{2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right). \tag{11}$$

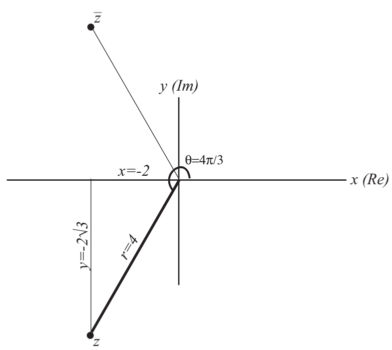
This gives

$$x = \Re(z) = \sqrt{2}\cos\frac{-\pi}{4} = 1 \tag{12a}$$

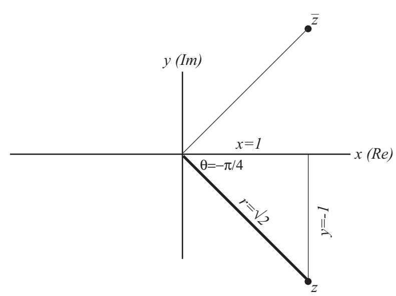
$$y = \Im(z) = \sqrt{2}\sin\frac{-\pi}{4} = -1 \tag{12b}$$

$$r = \sqrt{2}, \quad \theta = \frac{-\pi}{4} = -45° \tag{12c}$$

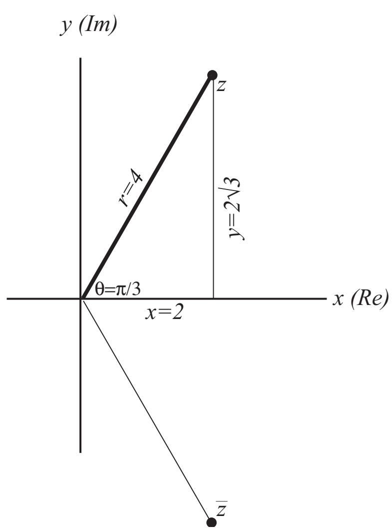
$$\bar{z} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right) \tag{12d}$$



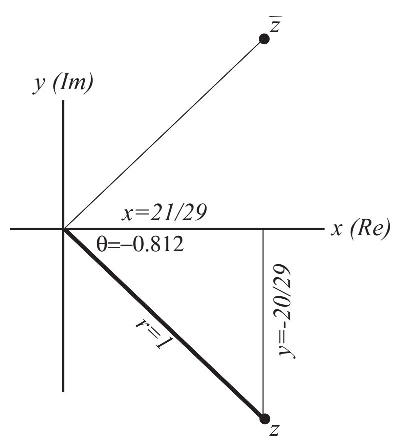
(a) Boas 2.4.12



(b) Boas 2.4.17



(c) Boas 2.5.5



(d) Boas 2.5.15

Figure 1: Problem 6

c) There are two approaches we could take to

$$z = \left(i + \sqrt{3}\right)^2. \tag{13}$$

The straightforward way is to simply square it,

$$z = \left(i + \sqrt{3}\right)^2 = i^2 + 2\sqrt{3}i + 3 = 2 + 2\sqrt{3}i, \tag{14}$$

and then the rest is just like the previous problem

$$x = \Re(z) = 2 \tag{15a}$$

$$y = \Im(z) = 2\sqrt{3} \tag{15b}$$

$$r = \sqrt{x^2 + y^2} = 4 \tag{15c}$$

$$\theta = \tan^{-1}\frac{2\sqrt{3}}{2} = \frac{\pi}{3} = 60^\circ \tag{15d}$$

$$\bar{z} = 2 - 2\sqrt{3}i \tag{15e}$$

The second way is to write $\sqrt{3} + i$ in polar form, $\sqrt{3} + i = 2e^{i\pi/6}$, so that

$$z = (2e^{i\pi/6})^2 = 4e^{i\pi/3}, \tag{16}$$

and then proceed by writing $z = 4(\cos(\pi/3) + i\sin(\pi/3))$.

d) In this problem we have a ratio of complex numbers,

$$z = \frac{5 - 2i}{5 + 2i}. \tag{17}$$

The first step is to eliminate the complex number in the denominator, and the way to do that is by multiplying by 1 in the form

$$1 = \frac{5 - 2i}{5 - 2i}. \tag{18}$$

This will simplify things because $5 - 2i$ is the complex conjugate of the denominator so the new fraction we get will have a real number in the denominator. Moving along,

$$z = \left(\frac{5 - 2i}{5 + 2i}\right)\left(\frac{5 - 2i}{5 - 2i}\right) = \frac{(5 - 2i)^2}{25 + 4} = \frac{25 - 20i - 4}{29} = \frac{1}{29}(21 - 20i), \tag{19}$$

and by now you know how these work out,

$$x = \Re(z) = \frac{21}{29} \tag{20a}$$

$$y = \Im(z) = -\frac{20}{29} \tag{20b}$$

$$r = \sqrt{x^2 + y^2} = 1 \tag{20c}$$

$$\theta = \tan^{-1}\left(\frac{-20/29}{21/29}\right) = \tan^{-1}\left(\frac{-20}{21}\right) = -43.6^\circ = -0.76 \tag{20d}$$

$$\bar{z} = \frac{1}{29}(21 + 20i). \tag{20e}$$