1. Recall that the delta function is an infinite peak at the position at which its argument is zero; as a result

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0). \tag{1}$$

(a)

$$\int_{-\infty}^{\infty} \cos x \delta x dx = \cos(0) = 1. \tag{2}$$

(b)

$$\int_{-\infty}^{\infty} x^2\delta(x - 2)dx = 2^2 = 4. \tag{3}$$

(c) The answer is 0 because the range of integration, from $x = 3$ to $x = 10$ does not include the point at which the delta function is non-zero ($x = 1$).

(d) We are to find

$$\int_{0}^{\infty} x\delta(5x - 1)dx. \tag{4}$$

Notice that we cannot use (1) because in that equation nothing multiplies the x inside the delta function. To get (4) into the right form make the change of variables $y = 5x - 1$ (so that $dx = dy/5$) to obtain

$$\int_{0}^{\infty} x\delta(5x - 1)dx = \frac{1}{5}\int_{-1}^{\infty} \frac{y + 1}{5}\delta(y)dy = \frac{1}{5}\frac{0 + 1}{5} = \frac{1}{25}. \tag{5}$$

An alternate solution uses the identity

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{6}$$

to factor the 5 out of the delta function given in the problem,

$$\delta(5x - 1) = \frac{1}{5}\delta(x - 1/5). \tag{7}$$

Writing the delta function this way also yields the answer $1/25$.

```
> restart;
```

Problem 2

There are really two parts to this problem: showing that the integral of the function given really is one, and showing that as σ approaches zero the funtion becomes a narrow spike.

First define the function...

```
> g:= exp(-x^2/sigma^2)/(sigma*sqrt(Pi));
```

$$g := \frac{e^{-\frac{x^2}{\sigma^2}}}{\sigma\sqrt{\pi}} \tag{1.1}$$

Now check the integral; note the assumption we have to place on σ.

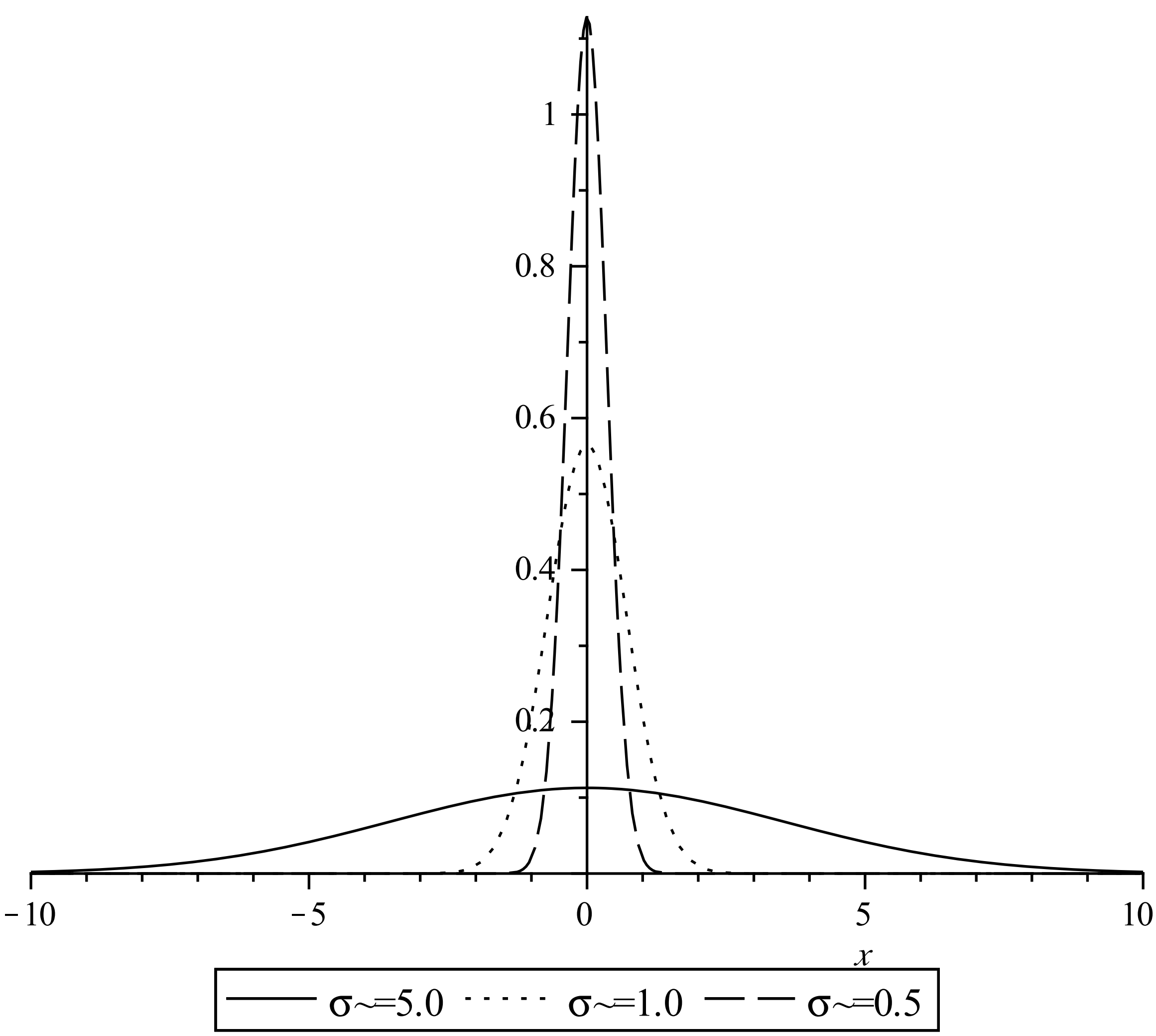
```
> assume(sigma>0): int(g,x=-infinity..infinity);
```

Hooray! We got 1 and the integral does not depend on σ.

$$1 \tag{1.2}$$

Next a graph to demonstrate that as σ decreases the peak becomes narrower and taller (one implies the other because the area is always 1).

```
> g:=(x,sig)->exp(-x^2/sig^2)/(sig*sqrt(Pi)):
> plot([g(x,5),g(x,1),g(x,0.5)],x,legend=[typeset(sigma,"=5.0"),
  typeset(sigma,"=1.0"),typeset(sigma,"=0.5")],linestyle=[solid,
  dot,dash],color=black,legendstyle=[location=bottom]);
```



1
0.8
0.6
0.4
0.2
−10
−5
0
5
10
x
σ~=5.0
σ~=1.0
σ~=0.5

2. The Fourier transform and its inverse are given by

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx, \qquad \text{Fourier transform,} \tag{8}$$

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha, \qquad \text{Inverse Fourier transform.} \tag{9}$$

   We are to show that these really are inverse operations. Substituting (8) for $g(\alpha)$ into the expression (9) gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')e^{-i\alpha x'}dx'e^{i\alpha x}d\alpha. \tag{10}$$

   Switching the order of integration and grouping the exponential terms together we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left(\int_{-\infty}^{\infty} e^{i\alpha(x-x')}d\alpha\right)dx', \tag{11}$$

   and

$$\int_{-\infty}^{\infty} e^{i\alpha(x-x')}d\alpha = 2\pi\delta(x-x') \tag{12}$$

   so that

$$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x-x')dx' = f(x), \tag{13}$$

   as we had hoped!