Physics 350 Midterm Examination Sample Solutions

Spring Semester 2009

1. In cosmology, we often represent the size of the Universe with a function called the scale factor, $a(t)$, where the scale factor of the universe today is $a(t_{now}) = 1$. In the late 1990s, observations of distant supernova confirmed that the universe was not only expanding (which has been known since the 1920s), but in fact the expansion is now accelerating! As this acceleration continues, the universe's expansion is best fit by the *deSitter model* where the scale factor should be:

$$a(t) = e^{Ht} \tag{1}$$

where H is a constant commonly called the Hubble constant.

(a) Find the Taylor series expansion of $a(t)$ about the point $t = 0$ (the Big Bang, the beginning of the Universe, is at $t = 0$), including the first three non-zero terms in the series.

The Taylor's series expansion can be written:

$$a(t) = \sum_{n=0}^{\infty} A_n \left(t - t_0\right)^n \text{ where } A_n = \frac{1}{n!} \left.\frac{d^n a\left(t\right)}{dt^n}\right|_{t=t_0}$$

where in this case, we are expanding around time zero ($t_0 = 0$) and so the first few terms here are:

$$\begin{aligned} A_0 &= \frac{1}{0!} e^{Ht}|_{t=0} = 1 \\ A_1 &= \frac{1}{1!} H e^{Ht}|_{t=0} = H \\ A_2 &= \frac{1}{2!} H^2 e^{Ht}|_{t=0} = \frac{H^2}{2} \end{aligned}$$

so the entire Taylor's series including the first three non-zero terms is:

$$a(t) \approx 1 + Ht + \frac{H^2}{2} t^2. \tag{2}$$

(b) Measurements of the Hubble constant today are mostly consistent with the observation that $\frac{da(t)}{dt} = Ha(t)$ called *Hubble's Law*. Use your Taylor's series approximation to work out $\frac{da(t)}{dt}$. Keeping only the first non-zero term in the Taylor's series approximation for $a(t)$ and $\frac{da(t)}{dt}$, can you verify Hubble's law is a consequence of deSitter's model?

Using the Taylor's series from equation 2 as suggested, I find:

$$\frac{da(t)}{dt} \approx H + H^2 t \approx H \tag{3}$$

and $a(t) \approx 1$ *such that Hubble's law,* $\frac{da(t)}{dt} = Ha(t)$ *holds. What does this mean, it means the deSitter model does predict the observed Hubble's law ... to first order.*

(c) What does the fact that Hubble's Law is observed to be reasonably accurate tell you about the age of the Universe today, t_{now}? Another way of answering this question is to answer "do you expect Hubble's law to hold in the distant future?" Your *explanation* of your answer is the key to full credit

Basically, we know that in fact if we go beyond first-order, equation 3 shows that $\frac{da(t)}{dt}$ *increases. That is the expansion rate of the Universe is increasing and the expansion is accelerating. This means Hubble's law will fail to hold once the second-order term in the Taylor's series becomes significant some time in the future (actually, we are seeing the effects of this now). The fact that the first-order model fits our observable Universe reasonably well means today, 13.7 Billion years after the Big Bang, the Universe is only at an early stage of its exponential expansion, where a first-order Taylor's series approximation is good enough.*

2. Write the number z in each part below in both Cartesian $(x+iy)$ form and in polar $(re^{i\theta})$ form. *Show your work!*

   (a) $z = (\sqrt{3}+i)(1-\sqrt{3}i)$

   This multiplication is easier to do in polar form (although if you prefer brute force, go for it). Since $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$, *I can state*

$$\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$$

   and since $\tan^{-1}\sqrt{3} = \frac{-\pi}{3}$ *then*

$$\sqrt{3}+i = 2e^{-i\frac{\pi}{3}}$$

   therefore

$$\begin{aligned} z &= (\sqrt{3}+i)(1-\sqrt{3}i) = 2e^{i\frac{\pi}{6}}2e^{-i\frac{\pi}{3}} \\ &= 4e^{-i\frac{\pi}{6}} = 4\cos\frac{\pi}{6} - 4\sin\frac{\pi}{6} = 2\sqrt{3}-2i \end{aligned} \quad (4)$$

   (b) $z = (i+1)^{32}$

   This problem is not really tractable unless you switch to polar notation, using

$$i+1 = \sqrt{2}e^{i\frac{\pi}{4}}$$

   such that

$$z = (i+1)^{32} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{32} = 2^{16}e^{i8\pi} = 65536 \quad (5)$$

(c) $z^3 = -8i$

This is another problem that is most easily tractable in polar notation, especially if you want to provide all three possible values for z*. Using* $-8i = 8e^{i\left(-\frac{\pi}{2}+2\pi n\right)}$ *where* n *is an integer then it is simple to state:*

$$
\begin{aligned}
z &= \left(8e^{i\left(-\frac{\pi}{2}+2\pi n\right)}\right)^{1/3} \\
&= 2e^{i\left(-\frac{\pi}{6}+\frac{2\pi}{3}n\right)} \\
&= 2e^{-i\frac{\pi}{6}}, 2e^{i\frac{\pi}{2}}, 2e^{i\frac{7\pi}{6}} \qquad (6) \\
&= \begin{cases} 2\cos\frac{\pi}{6} - 2i\sin\frac{\pi}{6} = \sqrt{3} - i \\ 2i \\ 2\cos\frac{7\pi}{6} + 2i\sin\frac{7\pi}{6} = -\sqrt{3} - i \end{cases} \qquad (7)
\end{aligned}
$$

Full credit on this problem requires you determine all three roots.

3. Although we talked about linear differential equations with constant coefficients primarily in the context of the classical simple harmonic oscillator, there are a couple of special cases of the Schrödinger equation from quantum mechanics that are also linear differential equations with constant coefficients. For example, the Schrödinger equation for a free particle is

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}E\psi(x) = 0, \tag{8}$$

where $\psi(x)$ is a function of x, and in general $\psi(x)$ may be complex. You may assume that E, the energy of the individual particles, and m, the mass of the individual particles are constants. $\hbar$ is the reduced Planck's constant (which as the name suggests, is a constant).

(a) Find the general solution for $\psi(x)$ that satisfies the Schrödinger equation for this case. This general solution will turn out to be the solution when you have a continuous stream of free particles.

This is a linear differential equation with the right hand side equal to zero, so we should first attempt a solution of the form $\psi(x) = Ae^{ax}$. *If we do, we find equation 8 becomes*

$$\begin{aligned}
\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}E\psi(x) &= 0 \\
a^2Ae^{ax} + \frac{2m}{\hbar^2}EAe^{ax} &= 0 \\
a^2 &= -\frac{2m}{\hbar^2}E \\
a &= \pm i\sqrt{\frac{2mE}{\hbar^2}} \qquad (9)
\end{aligned}$$

So we have two roots, thus the general solution to this Schrödinger equation is

$$\psi(x) = A_1 e^{+i\sqrt{2mE/\hbar^2}} + A_2 e^{-i\sqrt{2mE/\hbar^2}} \tag{10}$$

(b) The momentum of a particle with energy E is $p = \pm\sqrt{2mE}$. This quantity p should have shown up in your general solution. For the rest of this problem assume that the stream of particles represented by this solution is moving in the $+x$ direction (so it has only "positive" momentum). This is equivalent to saying that one of the two coefficients that shows up in the general solution is zero. What is the wave function of the particle in this case?

Noting that $p = \pm\sqrt{2mE}$ means that equation 10 can be re-expressed as

$$\psi(x) = A_1 e^{+ip/\hbar} + A_2 e^{-ip/\hbar}. \tag{11}$$

Noting that we are told that only the positive momentum solution should be considered, we can set $A_2 = 0$ and our final solution is:

$$\psi(x) = A_1 e^{+ip/\hbar} \tag{12}$$

.

(c) In quantum mechanics the absolute square of the wave function $|\psi|^2$ represents the probability of finding a particle at a particular position. What is $|\psi|^2$ for your answer to the previous part? Does it depend on position or not?

Remember that for complex numbers $|z| = \sqrt{z^* z}$ therefore $|z|^2 = z^* z$ and so in this case the probability density based on equation should be

$$|\psi(x)|^2 = A_1^* e^{-ip/\hbar} A_1 e^{+ip/\hbar} = A_1^* A_1. \tag{13}$$

Notice that the value of equation 13 is position independent, that is the particle has an equal probability of being anywhere.

Grading Note: *If you state the solution as just A_1^2 you will be partly incorrect because you must allow for complex coefficients. Similarly, the answer $A_1^2 e^{2ip/\hbar}$ is incorrect because the modulus, or absolute square, of a complex number is a real number.*

4. The Legendre functions appear in the solutions to a variety of problems in physics (notably problems in electricity and magnetism exploiting spherical coordinate systems). We discussed in class that the Legendre polynomials form an orthogonal basis for functions in the range -1 to 1. The first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}\left(3x^2 - 1\right) \\ P_3(x) &= \tfrac{1}{2}\left(5x^3 - 3x\right) \end{aligned}$$

   (a) Explain in words why it should make sense that the Legendre series approximation of $\sin(x)$ should contain only odd terms in the Legendre series. *Hint: No computations are necessary.*

   There are a variety of possible arguments. The easiest is to note that $\sin(x)$ *is odd/anti-symmetric across the interval* -1 *to* 1, *so it can only be fit by odd functions. A quick examination of the Legendre polynomials shows all the even number polynomials correspond to even functions and all the odd numbered polynomials correspond to odd functions, therefore it makes sense the Legendre series approximation for* $\sin(x)$ *should contain only odd terms.*

   (b) Verify that $P_1(x)$ and $P_3(x)$ are orthogonal to one another.

   To verify that $P_1(x)$ *and* $P_3(x)$ *are orthogonal to one another we simply need to confirm their inner product is zero.*

$$\begin{aligned} \langle P_1(x) | P_3(x) \rangle &= \int\limits_{x=-1}^{1} P_1(x) P_3(x) dx \\ &= \int\limits_{x=-1}^{1} (x) \left(\frac{1}{2}(5x^3 - 3x) \right) dx \end{aligned}$$

$$\begin{aligned}\langle P_1(x)|P_3(x)\rangle &= \frac{1}{2}\int\limits_{x=-1}^{1} 5x^4 - 3x^2 dx \\ &= \frac{1}{2}\left(x^5|_{x=-1}^{1} - x^3|_{x=-1}^{1}\right) \\ \langle P_1(x)|P_3(x)\rangle &= 0 \end{aligned} \tag{14}$$

(c) What are the first three ($\ell = 0, 1$, and 2) terms of the Legendre series approximation of the simple harmonic oscillator potential function $U(x) = \frac{1}{2}\kappa x^2$? Does this result make sense?

It's relatively easy to compute the terms in the Legendre series using the expression (from the equation sheet):

$$c_\ell = \frac{2\ell + 1}{2}\int\limits_{-1}^{1} F(x)P_\ell(x)dx \tag{15}$$

where $F(x) = U(x) = \frac{1}{2}\kappa x^2$ *here. Therefore the zeroth term is*

$$\begin{aligned} c_0 &= \frac{1}{2}\int\limits_{-1}^{1} (\frac{1}{2}\kappa x^2)P_0(x)dx \\ &= \frac{1}{4}\kappa\int\limits_{-1}^{1} x^2 dx \\ &= \frac{1}{4}\kappa\frac{1}{3}x^3|_{x=-1}^{1} \\ c_0 &= \frac{1}{6}\kappa, \end{aligned} \tag{16}$$

the first term is

$$\begin{aligned}
c_1 &= \frac{3}{2}\int\limits_{-1}^{1}(\frac{1}{2}\kappa x^2)P_1(x)dx \\
&= \frac{3}{4}\kappa\int\limits_{-1}^{1} x^3 dx \\
&= \frac{3}{4}\kappa\frac{1}{4}x^4|_{x=-1}^{1} \\
c_1 &= 0, \qquad (17)
\end{aligned}$$

and the second term is

$$\begin{aligned}
c_2 &= \frac{5}{2}\int\limits_{-1}^{1}(\frac{1}{2}\kappa x^2)P_2(x)dx \\
&= \frac{5}{8}\kappa\int\limits_{-1}^{1}\left(3x^4 - x^2\right)dx \\
&= \frac{5}{8}\kappa(\frac{3}{5}x^5|_{x=-1}^{1} - \frac{1}{3}x^3|_{x=-1}^{1}) \\
c_2 &= \frac{1}{3}\kappa, \qquad (18)
\end{aligned}$$

So the 2nd order Legendre series approximation of $U(x)$ *is*

$$\begin{aligned}
U(x) &\approx c_0P_0(x) + c_1P_1(x) + c_2P_2(x) \\
&\approx \frac{1}{6}\kappa P_0(x) + \frac{1}{3}\kappa P_2(x) \qquad (19)
\end{aligned}$$

This makes plenty of sense in that since $U(x)$ *is an even function, we would have expected* $c_1 = 0$*.*

(d) How close of an approximation is this Legendre series to the exact function $U(x)$? What can you say about the higher order terms in the Legendre series?

Inserting the expressions of $P_0(x)$ and $P_2(x)$ in equation 19 we have

$$\begin{aligned} U(x) &\approx \frac{1}{6}\kappa P_0(x) + \frac{1}{3}\kappa P_2(x) \\ &\approx \frac{1}{6}\kappa(1) + \frac{1}{3}\kappa\frac{1}{2}3x^2 - 1) \\ &\approx \frac{1}{6}\kappa + \frac{1}{2}\kappa x^2 - \frac{1}{6}\kappa \\ &\approx \frac{1}{2}\kappa x^2 \end{aligned} \tag{20}$$

So clearly equation 20 is identical to $U(x)$ which means all the higher order terms in the Legendre series approximation are zero! This was to be expected since for a simple polynomial like this, a perfect Legendre series approximation is possible if you go to the same order as the polynomial.