# Astrophysics Problem Set #5

Due Friday, October 2 at 4pm

In the homework problems below, "R&P" refers to your textbook. You may work together on this problem set, but all work presented here must be your own. **You must clearly acknowledge any people you collaborated with.**

**FIRST EXAM:** The First Midterm Exam is Tuesday, October 6. I have placed a study guide online.

**ASSUMED READING:** Before starting this homework, read R&P Chapter 5. You may also need to review your intro Physics textbook section on light waves.

1. **[R&P 5.4]** Molecules have additional degrees of freedom that atoms don't possess, namely, rotation and vibration. The energies associated with molecular rotation and vibration are quantized, and photons can be emitted or absorbed by molecules making transitions from one rotational or vibrational state to another.
    a. Show that the rotational energy of a system can be written as $E_{rot} = L^2/2I$ where $L$ is the angular momentum and $I$ is the moment of inertia.
    b. Suppose that angular momentum is quantized according to Bohr's hypothesis: $L = j\hbar$ , with $j$ being a positive integer. Consider the case of a diatomic molecule where the two atoms have equal mass $M$ (for instance, $H_2$, $O_2$, or $N_2$). Derive an expression for the rotational energy $E_{rot}$ in terms of $j$, $\hbar$, $M$, and $r_0$, the separation between the two atomic nuclei in the molecule.
    c. In the case of molecular hydrogen ($H_2$), which has $r_0 \approx 1$Å, estimate the wavelength of light produced by the $j = 2 \rightarrow 1$ rotational transition. Is this longer or shorter than the wavelength of visible light?

2. **[R&P 5.5 Tweaked]** This is a fairly simple problem about some energy transitions in the Sodium atom. In this problem, we compare the macroscopic properties of a gas, specifically temperature, to the microscopic properties of a gas, specifically kinetic energy and velocity of the atoms. These relationships are discussed in more detail on pages 124-126 of the textbook, but for our purposes you only need to know a few things.
    (1) From thermodynamics, we know that the distribution of particle speeds in a gas is described by the **Maxwell-Boltzmann distribution** (equation 5.40), which gives the fraction of particles in a given velocity bin based on temperature and particle mass.
    (2) If you change the Maxwell-Boltzmann distribution to be in terms of particle energy instead of velocity, you can find that the mean kinetic energy per particle is

$$\boxed{\langle E\rangle = \frac{3}{2}kT}$$ where $k$=1.381×10$^{-23}$ M$^2$ kg s$^{-2}$ K$^{-1}$ (the Boltzmann constant).

(3) Using this mean kinetic energy and assuming all the particles have the same mass, you can compute the root mean square velocity of the particles in the gas to be

$$\langle E\rangle = \frac{1}{2}m\langle v^2\rangle \longrightarrow \langle v^2\rangle = \frac{2\langle E\rangle}{m} = \frac{3kT}{m} \longrightarrow \boxed{v_{rms} = \langle v^2\rangle^{1/2} = \sqrt{\frac{3kT}{m}}}.$$

The root mean square velocity is a good estimate of the "typical" velocity of particles with a bulk temperature $T$.

Given this information and what you have learned about transitions in lecture, consider the following problems.

a. A neutral sodium atom has an ionization potential of $\chi = 5.1$ eV. What is the speed of a free electron that has just barely enough kinetic energy to collisionally ionize a sodium atom in its ground state? What is the speed of a free proton with just enough kinetic energy to collisionally ionize this atom?

b. What is the temperature $T$ of a gas in which the average particle kinetic energy is just barely sufficient to ionize a sodium atom in its ground state?

c. At the temperature $T$ computed in part (b), what is the expected thermal Doppler broadening, $\Delta\lambda/\lambda_0$, of a sodium spectral line? [**Hint:** The only stable isotope of sodium has mass number $A$ = 23, which has a mass per atom of $m$ = 23 amu × 1.67×10$^{-27}$ kg amu$^{-1}$ = 3.84×10$^{-26}$ kg. Using the same approach as on last week's homework, first show that:

$$\boxed{\frac{\Delta\lambda}{\lambda_0} \approx \frac{v_{rms}}{c}}$$

assuming that the difference between the highest velocity atoms in a gas cloud and the "systemic velocity" of the gas cloud is the rms velocity. Once you have done that, use that equation!]

3. **[R&P 5.7]** A slab of glass 0.2 meters thick absorbs 50% of the light passing through it. How thick must a slab of identical glass be in order to absorb 90% of the light passing through it? How thick must it be to absorb 99% of the light? How thick to absorb 99.9% of the light?

4. By applying the Boltzmann equation to the neutral hydrogen atom (neglect the possibility of ionization),
    a. Show that you can express the population of the $n^{th}$ energy level relative to the ground state at temperature $T$ as

    $$\frac{n_n}{n_1} = \frac{g_n}{g_1} e^{\frac{(157971K)\left[\frac{1}{n^2}-1\right]}{T}}.$$

    [**HINT:** Note that the energy of the neutral hydrogen electron in state $n$ can be written $E_n = (-13.6eV)\left[\frac{1}{n^2}\right] = (-2.18 \times 10^{-18} J)\left[\frac{1}{n^2}\right]$ and you are interested in the different in energy between a given energy level and the ground state.]
    b. Now, assuming that the statistical weight of each level is unity ($g_n$ = *1*), construct a graph showing your results for logarithm of the population of the $n^{th}$ energy level relative to the ground state, $\log_{10}\left[\frac{n_n}{n_1}\right]$, versus energy level $n$ for the Sun (assume $T$=5800K). (A computer program like *Microsoft Excel* may be useful here if you know how to use it)
    c. Do the same for a star with 20000K.
    d. What are the major differences between them?

5. To understand the relative important of the different parameters in the Saha equation, perform the following experiment. Rewrite the Saha equation shown in equation (5.69)

    $$\frac{n^{i+1}n_e}{n^i} = 2\frac{Q^{i+1}}{Q^i}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{\chi_i}{kT}\right)$$

    as

    $$\frac{n^{i+1}}{n^i} = \frac{\beta(kT)^{3/2}}{n_e}\exp\left(-\frac{\chi_i}{kT}\right) \quad \text{where } \beta = \left[2\frac{Q^{i+1}}{Q^i}\left(\frac{m_e}{2\pi\hbar^2}\right)^{3/2}\right]$$

    and for our purposes, $\beta$ is a constant for a particular atom. Assume that $T$=5000K, $n_e$ = $10^{15}$ cm$^{-3}$, and and $\chi_0$ =12 eV = 1.922×10$^{-18}$ J. By what factor does the ionization ratio, $n^1/n^0$ change when we separately do the following? **HINT:** Don't blinding start punching in numbers, first construct an expression for $(n^1/n^0)_{new}/(n^1/n^0)_{old}$ for each situation, eliminate as much as you can, then and only then, punch in the numbers.
    a. Double the temperature. Which is more important during the temperature change, the exponential term or the $T^{3/2}$ term?
    b. Double the electron density $n_e$.
    c. Double the ionization potential $\chi_0$.

6. Let $n_2^0$ be the number of hydrogen atoms in the first-excited state (with their electron in the second energy level) and $n_1^0$ be the number in the ground state. Consider hydrogen gas in a stellar atmosphere. Figure 5.13 in the textbook (shown to the right) shows the excitation ratio ($n_2^0 / n_1^0$) derived using the Boltzmann equation as well as the ionization fraction ($n^1/n^0$) derived using the Saha equation for hydrogen. In addition to these two ratios, we also have combination of the two effects, the fraction of all hydrogen atoms in the first-excited state ($n_2^0 / n$). Using Figure 5.13, find the excitation ratio ($n_2^0 / n_1^0$) and the excited fraction ($n_2^0 / n$) for each of the following stars:

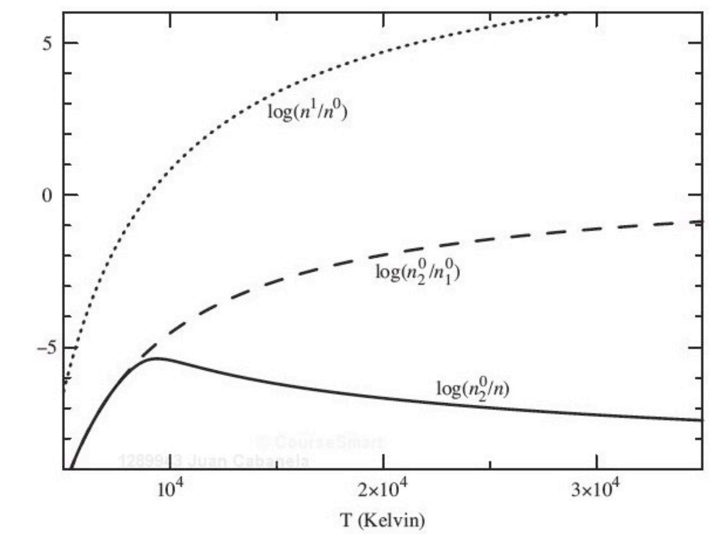


**FIGURE 5.13** Dotted line: the ratio $n^1/n^0$ from the Saha equation (a free electron density $n_e = 10^{20}$ m$^{-3}$ is assumed). Dashed line: the ratio $n_2^0/n_1^0$ from the Boltzmann equation. Solid line: the ratio $n_2^0/n$ from equation (5.73), that is, the fraction of hydrogen atoms that have a bound electron in the $n = 2$ state.

- The Sun, $T$= 5800K
- Vega, $T$=9600K
- Mintaka, $T$=30000K

Which star will exhibit the strongest Balmer absorption lines (recall, Balmer lines are due to electron transitions to/from the $n$=2 energy level in the Hydrogen atom)? **Explain your reasoning in arriving at this answer.**