Astrophysics Problem Set #3 Solutions
[40 points possible]

1. Explain (in words) how the Virial Theorem for gravitational systems
$2\langle K\rangle+\langle U\rangle=0$

allows you to use the redshift-measured velocities for galaxies in a galaxy cluster to determine the mass of the galaxy cluster. I am not asking you to actually do the calculation, but simply to explain essentially how you obtain $\langle K\rangle$ and $\langle U\rangle$ and how they are related to velocities and the masses.

[5 points possible] *In a nutshell,* $\langle K\rangle$ *is the time-averaged kinetic energy of the system and is related to the observed velocity of the galaxies in the galaxy cluster.* $\langle U\rangle$ *is the time average potential energy and in a gravitational system, it depends on the masses of the objects involved. Therefore the virial theorem should allow us to use the redshift-measured velocities for galaxies in a galaxy cluster to estimate the time-average kinetic energy. Once we have that, we can solve for the time-average potential energy which will be related to mass of the galaxy cluster.*

2. Every couple of years, an astrologer tries to argue for the scientific legitimacy of astrology (versus astronomy) by claiming it is the tides of the planets on the human body that affect a person's life (why the effects would depend on the tidal forces the moment you were born is unclear to me).
   a. Explain why we would expect the tidal forces on a human being less than 20 inches long at birth to be rather unimportant compared to tidal forces on the Earth. I want you to explain this in words, without resorting to explicit calculations. You may refer to formula if you have to, but concentrate on the relationships between variables, DO NOT CALCULATE ANYTHING.

   [3 points possible] *The only thing you need to know is that the tidal forces acting on an object depend linearly on its size (you can see this in equation 4.9 or equation 4.18). A human being, meters in size, is much smaller than a planet, we expect any tidal forces to be proportionally smaller.*

b. Consider my twin 6-1/2 year olds, Zakari and Carolina. They were born with a weight of about 1.7 kilos each and a length of about 43 cm each. Compute the <u>maximum</u> tidal forces on one of my twins due to:

[6 points possible] *For each of the following, all we need to know is that the tidal force can be computing using equation (4.8) in a more general form:*

$$\Delta F = \frac{2GM_{obj}m_{twin}R_{twin}}{a^3} = 2Gm_{twin}R_{twin}\frac{M_{obj}}{a^3}$$
$$= 2\left(6.67\times10^{-11}\frac{m^3}{kg\bullet s^2}\right)(1.7kg)(0.43m)\frac{M_{obj}}{a^3}$$
$$= \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{M_{obj}}{a^3}$$

So all I need is the mass and distance to each of the objects listed below and I can compute the tidal forces involved.

i. The moon. Assume it was at its average distance from Earth.

Assuming the moon is at its average distance from the Earth, then:

$$\Delta F = \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{M_{obj}}{a^3}$$
$$= \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{7.348\times10^{22}kg}{\left(3.844\times10^{8}m\right)^3}$$
$$= 1.26\times10^{-13}N$$

ii. The planet Jupiter. To maximize the tidal force, assume Jupiter was as close to the Earth as it can get (I leave it to you to explain how you determine what this closest approach distance is).

I have the mass of Jupiter from the last problem. As for its closest approach distance is its average orbital radius, 5.203 AU, minus earth's average orbital radius, 1.00 AU, that is 4.20 AU or 6.28×10^{11} *m. Using these numbers:*

$$\Delta F = \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{M_{obj}}{a^3}$$
$$= \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{1.90\times10^{27}kg}{\left(6.28\times10^{11}m\right)^3}$$
$$= 7.48\times10^{-19}N$$

So Jupiter's tidal force is much, much smaller than the Moon's . This makes sense because while it is much more massive than the moon, the effect of this increased mass is countered by an increased distance <u>*cubed*</u>*.*

iii. The obstetrician who delivered them, an 60 kilo female obstetrician. Assume an average distance of about 1 meter between her and my twins during the birth.

Given the relevant information for the obstetrician, her tidal force on one of my twin children can be computed as:

$$\begin{aligned}\Delta F &= \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{M_{obj}}{a^3}\\ &= \left(9.75\times10^{-11}\frac{m^4}{s^2}\right)\frac{60kg}{(1m)^3}\\ &= 5.85\times10^{-9}N\end{aligned}$$

So the tidal force exerted by the obstetrician is much, much greater than that exerted by any heavenly body.

c. What do the results of part (b) mean for the idea that planetary tides could significantly affect a birth, at least as far as scientifically "justifying" astrology is concerned?

[1 point possible] *In a nutshell, the astrologers should not try to rely on tidal forces to explain their astrological ideas since the tidal forces clearly show* <u>*who*</u> *helped birth your children has much more influence than any heavenly body. Something that is probably true, even discounting tidal forces. :)*

3. Let's consider the idea of "spaghettification" near a black hole. For our purposes, a "black hole" is just an extremely dense concentration of mass. As long as the local escape velocity near the black hole is not a significant fraction of the speed of light, we can use Newton's Law of Gravity to describe the gravitational effects near a black hole. Imagine you are falling into the black hole known as Cygnus X-1 orbiting the blue supergiant HDE 226868. Imagine the huge amount of X-ray radiation in this environment hasn't already killed you. This black hole has an estimated mass of 8.7 solar masses and an estimated diameter[1] of 26 km (as I said, very dense). Estimate how close you could get to the black hole, falling feet first, before being fatally "spaghettified" by the tidal forces. Since I am not an expert on human physiology, I will leave it up to you to clearly state how you estimated what would be a 'fatal' tidal force.

[10 points possible] *The exact distance of the fatal "spaghettification" will depend on the tidal force you assume will fatal. Let's call this force F_{dead}, All we need to tackle this problem is to figure out when the tidal force due to the black hole exceeds F_{dead}. As we did in the last problem, I can write a generalized expression for the tidal force due to the mass of the black hole, M_{BH}, at a distance, d, based on equation (4.8) of the book as.*

$$\Delta F = \frac{2GM_{BH}m_{me}R_{me}}{d^3}.$$

Setting this tidal force equal to F_{dead} and solving for the distance d we have the distance at which we are spaghettified

$$F_{dead} = \frac{2GM_{BH}m_{me}R_{me}}{d^3}$$

$$d = \left[\frac{2GM_{BH}m_{me}R_{me}}{F_{dead}}\right]^{1/3} = \left[2GM_{BH}\right]^{1/3}\left[\frac{m_{me}R_{me}}{F_{dead}}\right]^{1/3}$$

For Cygnus X-1, I have:

$$d = \left[2\left(6.67\times10^{-11}\frac{m^3}{kg\bullet s^2}\right)\left(8.7\times1.99\times10^{30}\,kg\right)\right]^{1/3}\left[\frac{m_{me}R_{me}}{F_{dead}}\right]^{1/3}$$

$$= 1.32\times10^7\frac{m}{s^{2/3}}\left[\frac{m_{me}R_{me}}{F_{dead}}\right]^{1/3}$$

So if I take the mass of a human body at m_{me} = 80 kg, the length at R_{me} = 1.75 m, and the fatal force at equivalent to a one ton weight (F_{dead} = 9800 N), then:

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[1] It's actually the radius, not the diameter, my bad...

$$d = 1.32\times10^{7}\,\frac{m}{s^{2/3}}\left[\frac{(80kg)(1.75m)}{9800N}\right]^{1/3}$$
$$= 1.32\times10^{7}\,\frac{m}{s^{2/3}}\left[1.43\times10^{-2}\,s^{-2}\right]^{1/3}$$
$$= 3.2\times10^{6}\,m$$

or about 3200 km from the black hole.

The escape velocity at this point is (equation 3.62)

$$v_{esc} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2\left(6.67\times10^{-11}\,\frac{m^3}{kg\bullet s^2}\right)\left(8.7\times1.99\times10^{30}\,kg\right)}{3.2\times10^{6}\,m}} = 2.7\times10^{7}\,\frac{m}{s}$$

which is actually 10% the speed of light. We may well have to consider General Relativistic effects this close to the black hole!

4. The X-rays in Cygnus X-1 (see last problem) are produced by the material falling into the black hole. Specifically, due to conservation of angular momentum, material doesn't fall directly into the black hole, instead it forms an accretion disk of material. Assume the material has been disrupted so that it is essentially a bunch of tiny particles orbiting the black hole in a 'gaseous' state.

a. Assuming circular orbits, explain in words what happens to the orbital velocity of material in the accretion disk as it approaches the black hole.

[4 points possible] *If we assume circular orbits (and Newton's Law of Gravity), then orbital velocity is*

$$v = \sqrt{\frac{GM}{r}}$$

so the orbital velocity goes as the inverse square root of the radius. This means as the radius decreases, the orbital velocity increases.

b. Using your answer from part (a), explain (in words) why material in the accretion disk drops into lower orbits over time. **HINT:** For it to drop into a lower orbit, it <u>must</u> lose energy. Think of some way that these particles can interact where there is a net lose in energy. **HINT #2:** Rub your hands together quickly, what is produced and why?

[3 points possible] *The material in a slightly higher orbit, which is revolving at a slower orbital velocity, will be 'passed' by material in a slightly lower (but faster) orbit. This means in essence there will be material 'rubbing' against other material, frictionally heating the* *material, converting some of the orbital kinetic energy into randomized kinetic energy (a.k.a. heat). This means the material loses energy and moves into a smaller orbit over time... all due to friction.*

c. Given you answer to part (b), explain why computing how fast the orbital speed varies as a function of radius will tell you where the accretion disk is hottest. Compute the necessary derivative to explain where the accretion disk will be the hottest. For reasons we will discuss later, this is where the X-rays in Cygnus X-1 are produced.

[3 points possible] *Given that the frictional heating of the disk depends on how fast material in adjoining orbits 'rub past' each other, the faster the orbital velocity varies with radius the higher the amount of heating. So if I take the orbital velocity*

$$v = \sqrt{\frac{GM}{r}}$$

then the derivative of this with respect to radius is

$$\frac{dv}{dr} = GM\frac{d}{dr}r^{-1/2} = -\frac{GM}{2}r^{-3/2}$$

which tells us that the frictional heating will increase as we get closer to the center of the accretion disk. The hottest part of the accretion disk is where the material is just about to fall into the black hole.

5. [R&P 4.2 Tweaked for context] Most of the exoplanets discovered so far have been Jupiter-sized or larger and in orbits smaller than 1AU! How close to the Sun could the planet Jupiter come without suffering tidal disruption? **NOTE:** We are ignoring the possibility of the heat from the Sun heating the planet enough to cause its gases to escape, so a Jupiter-sized planet outside the Roche limit might still be in for a destructive experience (I'll admit, I am not sure if the heat from the Sun is enough, maybe a future problem will explore this).

[5 points possible] *What we want to solve for here is the Roche limit of the Jupiter-Sun system. Based on equation (4.36) in the textbook, this should be:*

$$r_R = 2.44\left(\frac{\rho_\odot}{\rho_{Jup}}\right)^{1/3} R$$

which means I need to compute the densities of the Sun and Jupiter. Looking up the appropriate masses and radii and assuming these two objects are spherical:

$$M = \rho V = \frac{4}{3}\pi r^3 \rho \longrightarrow \rho = \frac{3M}{4\pi r^3}$$

$$\therefore$$

$$\rho_\odot = \frac{3M_\odot}{4\pi r_\odot^3} = \frac{3\left(1.99\times 10^{30}\,kg\right)}{4\pi\left(6.955\times 10^{8}\,m\right)^3} = 1.41\times 10^3\,\frac{kg}{m^3}$$

$$\rho_{Jup} = \frac{3M_{Jup}}{4\pi r_{Jup}^3} = \frac{3\left(1.90\times 10^{27}\,kg\right)}{4\pi\left(6.98\times 10^{7}\,m\right)^3} = 1.33\times 10^3\,\frac{kg}{m^3}$$

So the densities are almost the same! Computing the Roche limit using equation (4.36) as shown above, I have:

$$r_R = 2.44\left(\frac{1.41}{1.33}\right)^{1/3} R_\odot = 2.49 R_\odot$$

So within about 2.5 solar radii, Jupiter would be shredded by tidal forces. This is much, much less than an AU, so the exoplanets we have been seeing are not likely to be in danger of getting tidally disrupted.