Astrophysics Problem Set #2 Solutions
[XX points possible]

1. **[R&P 3.2 tweaked]** The asteroid Eros is seen in opposition from the Earth once every 847 days.
   a. What is the sidereal period of Eros?

   [4 points possible] *Since the orbital period of Eros seems likely to be larger than that of the Earth (e.g. - it is a "superior planet"), we can assume that Eros' orbit is larger than that of the Earth. As such, I can use equation (2.10) from the textbook to determine its sidereal period around the Sun based on the observed synodic period of 847 days:*

   $$\frac{1}{P_P} = \frac{1}{P_E} - \frac{1}{P_{syn}}$$

   $$\therefore P_{Eros} = \left[\frac{1}{365.256^d} - \frac{1}{847^d} \right]^{-1} = 642.2^d$$

   So the sidereal period works out to be 642.2 days.

   b. What is the length *a* of the semi-major axis of Eros' orbit?

   [3 points possible] *Given the sidereal period, it is a simple application of Kepler's Third Law (as described in equation 2.17 or 3.53 of the book). I will use the simpler expression (equation 2.17) where*

   $$P^2 = ka^3 \quad \text{where } k = 1.00 \frac{yr^2}{AU^3}$$

   to write an expression for the semi-major axis, a, given the sidereal period:

   $$P_{yr}^2 = a_{AU}^3$$

   $$a_{AU} = P_{yr}^{2/3}$$

   $$= \left(642.2^d \frac{1^{yr}}{365.256^d} \right)^{2/3} = (1.758)^{2/3}$$

   $$a_{AU} = 1.457$$

   So the semi-major axis of Eros is 1.46 AU, as we stated earlier, it is a "superior planet."

c. The eccentricity of the orbit of Eros is $e = 0.223$. Clearly show if Eros ever comes within 1AU of the Sun? Why is this an interesting question to consider?

[3 points possible] *Well, Eros has a semi-major axis of 1.46 AU. If its orbit is sufficiently eccentric, it could cross the Earth's orbit. Given that Eros is a rather large asteroid (its 11×11×34 km in size) would be very interesting to know if it crosses Earth's orbit, since it would present a much increased danger of collision between the asteroid and the Earth!*

Given the orbital eccentricity of 0.223 for Eros, the perihelion of Eros is at (the equation is stated on page 69, and I presented it in lecture):

$$\begin{aligned} q &= a(1-e) \\ &= 1.46AU(1-0.223) \\ &= 1.13AU \end{aligned}$$

So Eros does NOT cross Earth's orbit. We are safe from Eros.

2. There have been several recent missions to send spacecraft to the outer solar system. One of these missions is the *New Horizons* mission to Pluto, which launched in January 2006 and expected to pass by Pluto in July 2015.
   a. Estimate is the semimajor axis of the least-energy elliptical orbit of a space probe from Earth to Pluto (Pluto is near perihelion at a distance of about 31 AU) and how long would such a mission take?

   [2 points possible] *Since a least-energy orbit would start at Earth on one side of its major axis and end at Pluto at the other, the total length of the major axis would be 1AU + 31 AU = 32 AU. The semi-major axis of the least-energy elliptical orbit would be half this or 16AU.*

   b. How does this compare to the *New Horizons* mission flight time and what does this tell you?

   [3 points possible] *You can use Kepler's 3rd Law with units of AU and years, in which case:*

   $$P^2 = a^3 \Rightarrow P^2 = (16)^3 \Rightarrow P = (16)^{3/2} = 64 \text{ years}$$

   Cut this in half, since the flight time is half an orbit, you have a total flight time of ***32 years****! This tells you New Horizon's orbit, being faster than this, must have much higher energy than the "least energy" orbit.*

3. In order to achieve the speed necessary to get to Pluto so quickly, the *New Horizons* space probe engaged in a gravitational assist maneuver around Jupiter. Gravitational assist works because we use the gravity of the **moving** planet to gain energy relative to the Sun, thus speeding up the spacecraft relative to the Sun. Let's dissect this a bit:
   a. Imagine for the sake of argument that a spacecraft approached a planet "head on" as shown in the drawing to the right. Representing planetary parameters with capital letters and spacecraft parameters with lower case, show that taking into account conservation of kinetic energy (assuming same inbound and outbound "height" from the planet):

$$\frac{1}{2}MV_0^2+\frac{1}{2}mv_0^2=\frac{1}{2}MV_1^2+\frac{1}{2}mv_1^2$$

and conservation of momentum:

$$MV_0-mv_0=MV_1+mv_1$$

show that the velocity of the spacecraft after passing the planet must be $v_1 \approx 2V_0 + v_0 \approx 2V + v_0$ assuming $m \ll M$ and $V \approx V_0$.
HINT: This can be surprisingly nasty to do even though it is just algebra. This is really a "purely elastic collision" problem, in case you want to research it.
HINT 2: Try re-writing the conservation of energy and momentum conditions moving each objects velocities to one side and exploiting the algebraic equality $(a^2 - b^2) = (a+b)(a-b)$ to simplify things. Solve for V_1 and then use this to eliminate it and solve for v_1.

[5 points possible] *This is really just a nasty algebra problem and is in fact a purely elastic collision problem (where kinetic energy and momentum are both conserved). A direct approach by substituting one equation into the other and solving is painful. A much more elegant approach is to take the 2nd hint provided and rearrange the conservation of energy equation as:*

$$\frac{1}{2}MV_0^2-\frac{1}{2}MV_1^2=\frac{1}{2}mv_1^2-\frac{1}{2}mv_0^2$$
$$M\left(V_0^2-V_1^2\right)=m\left(v_1^2-v_0^2\right)$$
$$M\left(V_0-V_1\right)\left(V_0+V_1\right)=m\left(v_1-v_0\right)\left(v_1+v_0\right)$$

A similar approach with the conservation of momentum equations gives us:

$$MV_0 - MV_1 = mv_1 + mv_0$$
$$M\left(V_0 - V_1\right) = m\left(v_1 + v_0\right)$$

Now I can divide the conservation of energy equations by conservation of momentum equations in these forms to get and expression for V_1 to allow us to eliminate it later

$$\frac{M\left(V_0 - V_1\right)\left(V_0 + V_1\right)}{M\left(V_0 - V_1\right)} = \frac{m\left(v_1 - v_0\right)\left(v_1 + v_0\right)}{m\left(v_1 + v_0\right)}$$
$$\left(V_0 + V_1\right) = \left(v_1 - v_0\right)$$
$$V_1 = \left(v_1 - v_0\right) - V_0$$

And now I substitute this into the original conservation of momentum equation and solve for v_1:

$$MV_0 - mv_0 = MV_1 + mv_1$$
$$MV_0 - mv_0 = M\left(\left(v_1 - v_0\right) - V_0\right) + mv_1$$
$$\left(M + m\right)v_1 = MV_0 - mv_0 + Mv_0 + MV_0$$
$$\left(M + m\right)v_1 = 2MV_0 + \left(M - m\right)v_0$$

And finally, since we know M>>m, then M ≈ M+m ≈ M-m, this this expression becomes

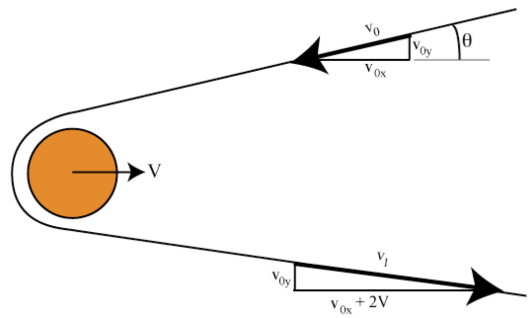
$$\boxed{v_1 = 2V_0 + v_0}$$

(Clearly several algebraic approaches are possible, some where m →0 work in a much more efficient manner. Any logic approach is acceptable.)

b. Considering part (a), how much does the speed (<u>not</u> velocity) of the space probe change in the reference frame of the planet? Surprised?

[2 points possible] *In the reference frame of the planet, it is at rest and the infalling object has speed $V+v_0$ and a final speed of $V+v_0$. The speeds are the same! This should make sense if energy is conserved in this interaction, since gravity is a conservative force.*

c. **[Extra Credit]** Let's do this same problem in two-dimensions (as shown in the figure to the right). In this case, only the component of the velocity parallel to the planet's motion is changed. Show that in a full 2-D case, the outbound <u>speed</u> is given by



2-D collision in "Solar System" Reference Frame

$$v_1 = (v_0 + 2V)\sqrt{1 - \frac{4v_0V[1-\cos\theta]}{(v_0+2V)^2}}$$

where θ is the angle of approach. [**HINT:** Despite being "extra credit," this problem is just a vector components problem. Apply the Pythagorean theorem to determine an expression for the total outbound velocity v_1 and work from there toward the expression above.]

[5 points Extra Credit possible] *Given the initial infall speed of* v_0*, we have vector components (as illustrated here):*

$$v_{0x} = v_0\cos\theta \qquad v_{0y} = v_0\sin\theta$$

and so the final outbound velocity components are:

$$v_{1x} = v_0\cos\theta + 2V \qquad v_{1y} = v_0\sin\theta$$

Which we can combine using the Pythagorean theorem to determine the total outbound speed:

$$\begin{aligned} v_1^2 &= (v_0\cos\theta + 2V)^2 + (-v_0\sin\theta)^2 \\ &= v_0^2\cos^2\theta + 4v_0V\cos\theta + 4V^2 + v_0^2\sin^2\theta \\ &= v_0^2 + 4v_0V\cos\theta + 4V^2 \\ &= v_0^2 + 4v_0V + 4V^2 + 4v_0V\cos\theta - 4v_0V \\ v_1^2 &= (v_0+2V)^2 + 4v_0V\cos\theta - 4v_0V \end{aligned}$$

Notice I used an algebraic trick of adding zero in the form $4v_0V - 4v_0V$ *to recover* $(v_0+2V)^2$ *on the right hand side. This gives me:*

$$v_1 = \sqrt{(v_0 + 2V)^2 + 4v_0 V\cos\theta - 4v_0 V}$$
$$= \sqrt{(v_0 + 2V)^2 + \frac{(v_0 + 2V)^2 4v_0 V[\cos\theta - 1]}{(v_0 + 2V)^2}}$$
$$= (v_0 + 2V)\sqrt{1 - \frac{4v_0 V[1 - \cos\theta]}{(v_0 + 2V)^2}}$$

Notice this reduces to v_0+2V if it's a head-on collision (θ=0°).

d. Now let's use the expression derived in part (c) in a real gravitational assist problem. Assuming the initial speed of the *New Horizons* space probe after its major fuel burns was 23 km/s and it approached Jupiter 'from the side' at an angle of 103° and the orbital velocity of Jupiter is 12 km/s. What should its speed be when on the outbound leg of its flight after the gravitational assist?

[3 points possible] *This is just a plug and chug with the numbers given using the equation derived in part (c):*

$$v_1 = (v_0 + 2V)\sqrt{1 - \frac{4v_0 V[1 - \cos\theta]}{(v_0 + 2V)^2}}$$
$$= \left(23\tfrac{km}{s} + 2\left(12\tfrac{km}{s}\right)\right)\sqrt{1 - \frac{4\left(23\tfrac{km}{s}\right)\left(12\tfrac{km}{s}\right)[1 - \cos(103^\circ)]}{\left(\left(23\tfrac{km}{s}\right) + 2\left(12\tfrac{km}{s}\right)\right)^2}}$$
$$= 29.3\tfrac{km}{s}$$

So in this case, the New Horizons gained about 6 km/s of speed for "free".

4. **Journal Reading:** Please download "On the Masses of Nebulae and of Clusters of Nebulae" (Zwicky, F. 1937, Astrophysical Journal, 86, 217) from the electronics handouts page on our course website. The article discusses various methods used by Fritz Zwicky to determine the masses of individual galaxies and clusters of galaxies (referred to at the time as "nebulae") . This paper saw the application of the virial theorem to the problem. It is a longish paper (29 pages), so I don't expect you to learn all the material in it in one pass, but pay attemtion especially to the sections on the application of the viral theorem to the problem and the results. Write up of 2 questions and your best attempt at answers based on the paper. Turn them in with this problem set.

   [10 points possible] *This problem will be graded based on the depth of your two questions and the thoroughness of your solutions to the two questions you propose.*