Astrophysics Problem Set #1 Solutions
[50 points possible]

1. **[R&P 2.2 modified]** On 2003 August 27, Mars was in opposition as seen from the Earth. On 2005 July 14 (687 days, a.k.a. a Martian Sidereal Period, later), Mars was in western quadrature as seen from the Earth. What was the distance of Mars from the Sun on these dates, measured in astronomical units (AU)? How does this distance compare to the semimajor axis length of the Martian orbit (see Table A.3 attached to this homework for semimajor axis)? What does this mean?

   You may assume the Earth's orbit is a perfect circle. [**Big Hint:** The sidereal period of Mars is also 687 days. Use this information and make a drawing of where the planets are on 2003 August 27 and 2005 July 14.]

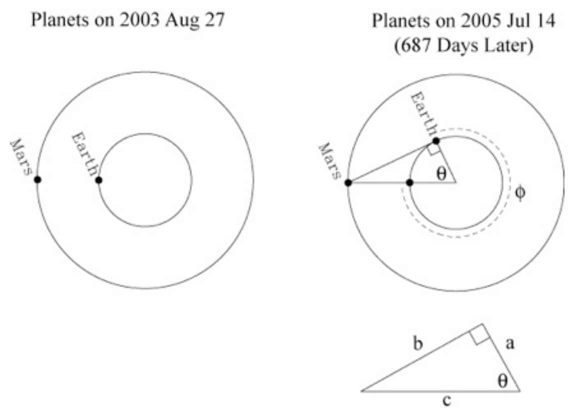


Figure 1: This figure to the left shows the Earth and Mars when Mars was on opposition on 2003 Aug 27. To the right, we see these planets one Martian sidereal period later, when we are told Mars is in quadrature. . Below that sketch I break out the right triangle I drew, where *a* is the Earth–Sun distance, *c* is the Mars–Sun distance, and *b* is the Earth–Mars distance when Mars is at western quadrature.

[10 points possible] *In making the sketches to the right, I note that during 687 days, Mars revolves around the Sun once (since 687 is Mars' sidereal period) whereas the Earth makes*

$$N_{orbit} = \frac{687 \text{ days}}{365.256 \text{ days/orbit}} = 1.881 \text{ orbits}$$

around the Sun. I used that and the fact that the Mars-Earth-Sun angle is 90˚ at quadrature to make the drawing to the right. Below that I sketch the right triangle of interest. I know the Earth-Sun distance (labelled "a") is 1AU, so if I can determine the angle θ I can determine the length of side "c", which corresponds to Mars' distance from the Sun.

I can determine the angle θ by noting that the difference in angle travelled by Earth and Mars during this time period (labelled φ) is just

$$\phi = (\omega_E - \omega_M)\tau = \left(\frac{2\pi}{P_E} - \frac{2\pi}{P_M}\right)\tau$$

(this follows the equations developed in the discussion on page 45-46 of R&P). In

this case, the time elapsed, $\tau = P_M = 687$ *days, therefore I can simplify this as:*

$$\phi = \left(\frac{2\pi}{P_E} - \frac{2\pi}{P_M}\right)P_M = 2\pi\left(\frac{P_M}{P_E} - 1\right)$$

$$= 2\pi\left(\frac{687^d}{365^d.256} - 1\right)$$

$$= 2\pi(0.88087) = 5.53469^{rad}$$

$$= 317^\circ.11$$

So Earth is 317˚.11 ahead of Mars. This means the Earth-Sun-Mars angle is:
$\theta = 360^\circ - \phi = 42^\circ.89$.

Now, simple trigonometry gives the distance of Mars from the Sun,

$$c = \frac{a}{\cos\theta} = \frac{1AU}{0.733} = 1.365AU$$

which is less than the length of the semimajor axis of the orbit of Mars, listed in Table A.3 as 1.524 AU. ***This tells us that the orbit of Mars cannot be circular.***

If you are careful, you could compute the perihelion and aphelion distances of Mars using the eccentricity in Table A.3 of 0.0934 to say:

$$r_p = q = a(1-e) = 1.382AU$$

$$r_a = Q = a(1+e) = 1.666AU$$

which actually excludes the distance we computed. I can't be certain, but I suspect this is because we assumed Earth's orbit was circular, leading to a small miscalculation of Mars' distance from the Sun.

2. **[R&P 2.3 with a hint]** In the 1670s, the astronomer Ole Røemer observed eclipses of the Galilean satellite Io as it plunged through Jupiter's shadow once per orbit. He noticed that the time between observed eclipses became shorter as Jupiter came closer to the Earth and longer as Jupiter moved away. Roemer calculated that the eclipses were observed 17 minutes earlier when Jupiter was in opposition than when it was close to conjunction. This was attributed by Røemer to the finite speed of light. From Røemer's data, compute the speed of light, first in AU min^{-1}, then in m s^{-1} . [**HINT:** This question is really simple if you understand the definitions of "opposition," "conjunction," and "AU."]

   [5 points possible] *The difference in Jupiter's distance from Earth during opposition and conjunction is simply the diameter of the Earth's orbit, D = 2 AU. The speed of light is thus c = 2 AU/17 min = 0.118 AU/min. In SI units, this becomes*

   $$c = \frac{0.118 AU}{min} \times \frac{1.49\times10^{11} m / AU}{60 s / min} = 2.92\times10^{8}\frac{m}{s}$$

   [The solution I present here is the version from the Instructor's Solution Manual with some edits.]

3. Using orbital data for Earth's Moon (reproduced to the right), find the mass of the Earth. Compare it to the value listed in Table A.3 attached to this problem set. Can you explain any difference? **(HINT:** If necessary, you can safely assume the Moon's mass is much less than the Earth's mass.)

   Orbital Parameters for the Moon
   (from Table A.4 of R&P)

Orbital Radius:	384,400 km
Orbital Period:	27.32 days
Radius of World:	1737 km
Mass:	7.348×10^{22} kg

   [5 points possible] [Grading Note: I had assumed you would have covered Newton's form of Kepler's Third Law in lecture before the homework was due. I'll be generous in grading this one since that was not the case.] *Newton's form of Kepler's Third Law (equation 3.53) can be re-written:*

   $$P^2 = \frac{4\pi^2 a^3}{G(M+m)}$$

   $$M + m = \frac{4\pi^2 a^3}{GP^2}$$

   If we assume the Moon is much less massive than the Earth, then M≫m and

$$M \approx \frac{4\pi^2 a^3}{GP^2} = \frac{4\pi^2\left(3.844\times10^8\,m\right)^3}{\left(6.67\times10^{-11}\,\frac{N\bullet m^2}{kg^2}\right)\left(27.32d\times\left(\frac{86400s}{1d}\right)\right)^2}$$

$$= 6.03\times10^{24}\,kg$$

Earth's mass is listed in the footnote of Table A.3 as 5.974×10^24^ kg. So we overestimated the mass a bit (about 1%), which makes sense, given the Moon's mass is about 1% of Earth's (think about that and be sure you can explain why, maybe it will be on the midterm).

4. A low Earth orbit for a satellite (or the Space Shuttle), no more than a few hundred kilometers over the surface of the Earth, has an orbital period of 90 minutes. Interestingly, the *Messenger* spacecraft, scheduled to enter a low orbit around Mercury in March 2011, also has an orbital period of about 90 minutes. The orbital periods of the spacecraft in low Martian orbit are also in the neighborhood of 90 minutes (actually just a bit longer). These are objects with radically different masses. Why would all these orbits have similar periods?

[Grading Note: I had assumed you would have covered Newton's form of Kepler's Third Law in lecture before the homework was due. I'll be generous in grading this one since that was not the case.]

a. What physical characteristics do the Earth, Mercury, and Mars all have roughly similar values of despite their differing masses? **HINT:** Orbits depend on gravity and gravity depends on mass. So this physical characteristic must be related to mass. **HINT #2:** Consider what is similar about the Earth, Mercury, and Mars and see if you can use the data in Table A.3, attached to this problem set, to compute the corresponding physical characteristic. Reviewing Chapters 8.1 to 8.2 of the textbook might give you a few hints as to which physical characteristic to consider.

Planet	Mean Radius (Earth = 1)	Mass (Earth =1)	Density (Earth=1)
Mercury	0.383	0.0553	0.984
Earth	1	1	1.000
Mars	0.532	0.1074	0.713
Jupiter	10.97	317.8	0.241

***[4 points possible]** The key physical property related to mass that is similar for all these worlds is that they are made up of rock and metal (mostly rock), so the values of their **average density**, ρ = M/V, are very similar. The values I computed for these densities are shown in the table. Notice that Mercury and Earth in particular are very similar in density.*

Other answers that have some basis in logic (i.e. – they are related to mass, but not mass) will be considered)

b. Derive the relationship between orbital period of a "low orbit" and this physical characteristic from part (a) assuming a "low orbit" essentially has an semimajor axis equal to the planet's radius and the satellite has much lower mass than the planet.

[4 points possible] *Using Newton's version of Kepler's 3rd Law, we have*

$P^2 = \frac{4\pi^2}{G(M+m)} a^3$. *If we assume M≫m we can state*

$$P^2 = \frac{4\pi^2}{GM} R^3$$

Furthermore, if we take a "low orbit" to mean a ~ R and that the planet is a uniform sphere, such that the mass is just

$$M = V\rho = \frac{4}{3}\pi R^3 \rho$$

Then we can combine these two equations to state

$$P^2 = \frac{4\pi^2}{G\left(\frac{4}{3}\pi R^3 \rho\right)} R^3 = \frac{3\pi}{G\rho}$$

$$\therefore P = \sqrt{\frac{3\pi}{G\rho}}$$

c. Given your derivation in part (b), should a low Jovian orbit around Jupiter have a period higher or lower than 90 minutes.

[2 points possible] *Given that $P \propto \rho^{-1/2}$ from part (b) and that Jupiter has a lower density than Earth, then the orbital period of a low orbit over Jupiter should be longer for Jupiter. This is also the case for Mars, which has a lower density, as I mentioned in the setup for part (a).*

5. **[R&P 3.1 modified]** Comet Hale–Bopp has an orbit about the Sun with eccentricity e=0.9951 and semimajor axis length a = 186.5 AU.
   a. Try to accurately sketch the orbit, labeling the Sun and the perihelion and aphelion of Hale-Bopp's orbit. Get the axial ratio, b/a, right! For a sense of scale, note that Neptune's orbit approximately 30 AU in radius. [**HINT:** Figure 3.6 might help you relate perihelion and aphelion distances to a and e. Attending class might help too!]

      [5 points possible] *Starting with equation (3.36), I can derive the axial ratio between the semimajor axis, a, and the semiminor axis, b, of an orbit with an eccentricity, e, to be:*

      $$e = \left(1 - \frac{b^2}{a^2}\right)^{1/2} \rightarrow \frac{b^2}{a^2} = 1 - e^2$$

      $$\therefore \frac{b}{a} = \sqrt{1-e^2} = \sqrt{1-0.9951^2} = 0.09887$$

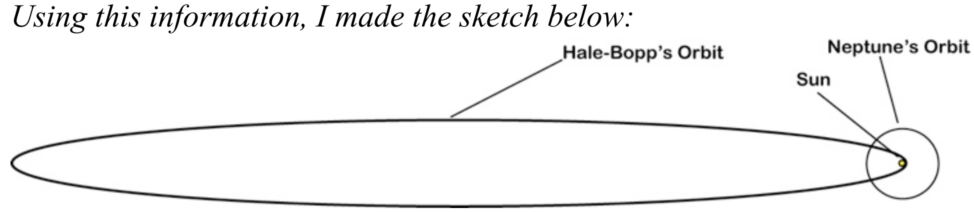
      So the semiminor axis is approximately 1/10 the length of the semimajor axis! This is one squashed elliptical orbit.

      By examining Figure 3.6, we can see that at the perihelion (closest approach), the distance between the object and the Sun must be:
      q = (1-e)a = (1 − 0.9951)186.5 AU = 0.91 AU.

      A similar argument let's you state that at aphelion, the distance between the object and the Sun must be:
      Q = (1+e)a = (1.9951)186.5 AU = 372.1 AU.

      This gives a total major axis length of 372.1+0.91 = 373 AU!

      Using this information, I made the sketch below:



   b. What is the sidereal orbital period of Comet Hale–Bopp? What is Comet Hale–Bopp's distance from the Sun at perihelion? What is its distance from the Sun at aphelion? Comet Hale–Bopp passed through perihelion in 1997 April 1. When did the previous perihelion passage of Comet Hale–Bopp occur?

***[5 points possible]** Using Kepler's Third Law (equation 2.17),*

$P^2 = Ka^3$ *where for an orbit around the Sun* $K = 1\ yr^2\ AU^{-3}$

we can state that for Hale-Bopp, which has a=186.5 AU, the period is

$$P_{yr} = a_{AU}^{3/2} = (186.5)^3 = 2547$$

or 2547 years. This means its previous perihelion passage was in

1997-2547 = -550

or 549 BC (Asking why "549 BC"? Remember there was no year zero!)!

6. **[R&P 3.6 tweaked]** Communications and weather satellites are often placed in *geosynchronous* orbits. A geosynchronous orbit is an orbit about the Earth with orbital period P exactly equal to one sidereal day (which is 86160 seconds, or about 4 minutes less than a solar day). What is the semimajor axis a_{gs} of a geosynchronous orbit? What is the orbital velocity v_{gs} of a satellite on a circular geosynchronous orbit?

[Grading Note: I had assumed you would have covered Newton's form of Kepler's Third Law in lecture before the homework was due. I'll be generous in grading this one since that was not the case.]

***[10 points possible]** Since I know the orbital period (86160 seconds), I can determine the semimajor axis of this orbit using Kepler's Third Law. I will have to use Newton's version of Kepler's Third Law, equation (3.53) where I can take the mass of the satellite to be insignificant compared to the Earth, so m≪M and so:*

$$P^2 = \frac{4\pi^2}{GM}a^3 \rightarrow a = \left[\frac{GM}{4\pi^2}P^2\right]^{1/3}$$

and therefore, just "plugging and chugging"...

$$a = \left[\frac{\left(6.67\times10^{-11}m^3kg^{-1}s^{-2}\right)\left(5.974\times10^{24}kg\right)}{4\pi^2}\left(86160s\right)^2\right]^{1/3}$$

$$= \left[7.493\times10^{22}m^3\right]^{1/3}$$

$$= 4.22\times10^7 m = 42200 km$$

Taking the orbit to be circular (or just using equation 3.74, which assumes this), we have an orbital speed of:

$$v_{orbit} = \frac{2\pi a}{P} = \frac{2\pi\left(4.22\times10^7 m\right)}{86160s} = 3.07\times10^3\frac{m}{s} = 3\frac{km}{s}$$