Astrophysics 410 Midterm #1 Exam Solution Key

Fall Semester 2009

1. [**30 points total**] Some conceptual questions to get your astrophysical juices pumping...

   (a) [**10 points**] When Newton developed his explanation of Kepler's Laws of Planetary Motion in terms of his three laws of Physics and law of Gravitation, he not only provided a physical justification for the laws, but also 'extended' their power quite a bit. Give an clearly explained example of how Newton, by explaining one of Kepler's Laws in physical terms, also extended its application well beyond where Kepler originally intended.

   Newton's explanation of Kepler's Laws of Planetary motion led to the realization that

   - *Kepler's Laws don't just apply to planets, but apply to any two objects orbiting each other under the influence of gravity.*
   - *Kepler's First Law that planets orbit in ellipses with the Sun at one focus just represented a subset of possible orbits. In addition to the closed elliptical orbits, objects can orbit other objects in open parabolic or hyperbolic orbits.*
   - *Kepler's Third Law, when derived using Newton's Physics, shows that there is indeed a relationship between the period of an orbit squared and the semi-major axis of the orbit cubed, but the constant of proportionality between these two depends on the masses of the two objects involved. This has made Kepler's Third Law immensely powerful to astronomers, since in essence it allows us to make an estimate of the mass of any object in space as long as we can spot something else orbiting it (and we can estimate the period and semi-major axis of the orbit).*

   I will accept any of these answers as a reasonable interpretation of Newton's extension of Kepler's Laws of Planetary Motion.

(b) [**10 points**] Explain why we can't simply use an equation just like the Bohr model of the Hydrogen atom to describe all neutral atoms. In other words, what assumption made in the Bohr model of the atom doesn't hold for multi-electron neutral atoms and why?

In Bohr's model of the hydrogen atom, the electron is held in its orbit around the proton by electrical forces such that the electric potential energy is what gives the electron its energy. The problem in multi-electron atoms is that the electron not only interacts electrically with the protons in the nucleus, but also with other electrons in the atom. And while the nucleus can be treated as mostly motionless, the electrons' positions relative to each other are constantly changing and thus the expression for the electric potential energy of the electrons is ***NOT*** *a simple* $1/r$ *expression, but rather some ever-varying sum of such potentials. Ugly to deal with!*

(c) [**10 points**] In his recent trade paperback, *Death from the Skies!*, Phil Plait describes various astronomical phenomena that could conceivably lead to the end of life on Earth. One of the phenomena he considers is the destruction of the Earth by the passage of a 10 solar mass black hole through the solar system. When the black hole is 7 million miles from Earth, the Earth is destroyed... long before we can be 'sucked into' the black hole. In fact, Earth never falls into the black hole, but Earth is destroyed. Explain how the black hole could destroy us at such a distance.

Actually, this one is fairly simple. What destroys the Earth is the differential gravitational forces (a.k.a. tidal forces) of the black hole's gravity across the surface of the Earth. In essence, once the difference in the pull of the black hole's gravity on the side closest to the black hole versus the opposite side of the earth exceeds $2g$ *where* g *is the traditional* $9.8m/s^2$*, then the earth will be torn apart since the tidal forces of the black hole exceed the Earth's self-gravity.*

2. [**35 points total**] Astronomers at the Spacewatch telescope on Kitt Peak discover an asteroid with a synodic period of 1.5 years.

   (a) [**10 points**] Explain, in English, the difference between the synodic period and the sidereal period of the asteroid.

   The synodic period is the time it takes the asteroid (or any other object) to appear in the same place as seen from the Earth relative to the Sun. Easiest way to express this is that it is the time between two successive conjunctions of the object with the Sun as seen from Earth. The sidereal period is the time it takes the asteroid (or any other object) to complete one full 360° *orbit around the Sun.*

   (b) [**10 points**] What are the two possible values for the sidereal period of this asteroid? *Clearly explain your reasoning!*

   The conversion from synodic period to sidereal period depends on whether we believe the object is in an inferior orbit to the Earth (i.e. - semi-major axis smaller than Earth's) or a superior orbit to the Earth (i.e. - semi-major axis larger than Earth's). If the orbit is smaller, than the angular velocity of the object is greater than that of Earth and we can say:

$$\omega_P = \omega_E + \omega_{syn} \Longrightarrow \frac{1}{P_P} = \frac{1}{P_E} + \frac{1}{P_{syn}}. \tag{1}$$

   If the orbit is larger, than the angular velocity of the object is smaller than Earth's and

$$\omega_P = \omega_E - \omega_{syn} \Longrightarrow \frac{1}{P_P} = \frac{1}{P_E} - \frac{1}{P_{syn}}. \tag{2}$$

   Using equations **??** *and* **??** *we can work out the possible sidereal* *periods of this asteroid to be:*

$$\begin{aligned}\frac{1}{P_P} &= \frac{1}{P_E} \pm \frac{1}{P_{syn}}. \\ &= \frac{1}{1.0^{yr}} \pm \frac{1}{1.5^{yr}} \\ &= \frac{5}{3^{yr}}, \frac{1}{3^{yr}} \\ P_P &= \frac{3}{5} yr, 3yr \qquad (3)\end{aligned}$$

Therefore $\boxed{\textit{the sidereal period of this asteroid is either 0.6 years or 3.0 years}}$.

(c) [**5 points**] What are the two possible values for the semi-major axis of the asteroid's orbit? *Again, clearly explain your reasoning!*

Given the (sidereal) orbital periods from part (a), we can use Kepler's third law to determine the possible semi-major axes. Using period in years and semi-major axis in AU, then Kepler's Third Law is just

$$P_{yr}^2 = a_{AU}^3 \qquad (4)$$

therefore

$$a_{AU} = P_{yr}^{2/3}. \qquad (5)$$

And therefore the two possible semi-major axes are

$$a_{AU} = \begin{cases} 0.6^{2/3} = 0.71 \\ 3^{2/3} = 2.08 \end{cases} \qquad (6)$$

So the semi-major axes of this asteroid's orbit is either $\boxed{\textit{0.71 AU or 2.08 AU}}$.

(d) [**10 points**] Imagine NASA decides to launch a probe to the asteroid. Assuming, like most asteroids, that it is in a superior orbit to the Earth, compute the additional velocity the rocket would need (beyond the 30 km/s of Earth's orbital velocity) in order to reach the asteroid in a least-energy orbit. You may assume circular orbits [for Earth and the asteroid] to make the math simpler.

[Since the part of the question in brackets' above was

not in the original question, I was generous grading for those who tried to make the transfer orbit circular.] *The least-energy transfer orbit would be one with perihelion at about 1.0AU (earth's orbit) and aphelion at the asteroid's orbit of 2.08 AU, so the semi-major axis of the transfer orbit would be* $a = \frac{1}{2}(1.00AU + 2.08AU) = 1.54AU$. *To determine the velocity, we need to use the vis viva equation*

$$v^2 = G\left(m_1 + m_2\right)\left[\frac{2}{r} - \frac{1}{a}\right], \tag{7}$$

where in this case, $m_1 = M_{Sun}$ *and* $m_2 = M_{asteroid}$ *which means* $m_1 + m_2 \approx M_{Sun}$. *Furthermore,* $r = 1.0AU$ *and the semi-major axis of the transfer orbit is* $a = 2.0AU$. *Using this, we can use the vis viva to give us the injection velocity*

$$\begin{aligned} v^2 &= \left(6.674 \times 10^{-11}\frac{N \cdot m^2}{kg^2}\right)\left(2.0 \times 10^{30}kg\right) \qquad (8) \\ &\quad \left[\frac{2}{1.496 \times 10^{11}m} - \frac{1}{1.54 \times 1.496 \times 10^{11}m}\right], \\ &= 1.21 \times 10^9 m^2/s^2 \\ v &= 3.47 \times 10^4 m/s \qquad (9) \end{aligned}$$

So we need an injection velocity of 34.7 km/s, which means the probe must leave Earth moving at 4.7 km/s (assuming Earth is moving at 30.0 km/s and we take advantage of this orbital speed to ease up our work).

3. [**35 points total**] In 1982 the first **millisecond pulsar** was discovered. Millisecond pulsars are believed to be neutron stars in binary star systems in which the extremely compact neutron star can accrete material from a very large red supergiant star, causing the pulsar to "spin up." Let's explore the feasibility of this model by considering a binary star system consisting of a 2 $M_\odot$ neutron star with a 2 $M_\odot$ companion red supergiant star. **NOTE:** You can answer part (c) of this problem without answering parts (a) and (b).

   (a) [**15 points**] Mass transfer will start between these two stars when the tidal forces due to the gravity of the neutron star start removing the outer layers of the Red Giant, in other words, when the red giant falls within its Roche limit radius of the neutron star. Determine what the separation between these two stars is when mass transfer begins. In order to do this, you will need to estimate the density of the two stars. You can assume the companion red supergiant star has a radius of about 1 AU whereas the neutron star has a radius of about 10 km.

   Both stars have masses of 2 $M_\odot \approx 4 \times 10^{30}$ *kg, the only difference is their radius. A neutron star has a radius of about 10 km so the mean density must be:*

$$\rho_{NS} = \frac{M_{NS}}{\frac{4}{3}\pi r_{NS}^3} = \frac{4 \times 10^{30} kg}{\frac{4}{3}\pi (10^4 m)^3} = 9.5 \times 10^{17} \frac{kg}{m^3} \tag{10}$$

   whereas the Red supergiant is a lot more diffuse, with a radius of 1AU and thus a mean density of:

$$\rho_{RS} = \frac{M_{RS}}{\frac{4}{3}\pi r_{RS}^3} = \frac{4 \times 10^{30} kg}{\frac{4}{3}\pi (1.5 \times 10^{11} m)^3} = 2.8 \times 10^{-4} \frac{kg}{m^3} \tag{11}$$

   Therefore $\rho_M/\rho_m = \rho_{NS}/\rho_{RS} = 9.5 \times 10^{17} \frac{kg}{m^3} / 2.8 \times 10^{-4} \frac{kg}{m^3} = 3.3 \times 10^{21}$*. [If you were clever, you might notice I set up the two masses to be equal, therefore*

$$\frac{\rho_M}{\rho_m} = \frac{\rho_{NS}}{\rho_{RS}} = \frac{2M_\odot/\frac{4}{3}\pi r_{NS}^3}{2M_\odot/\frac{4}{3}\pi r_{RS}^3} = \left(\frac{r_{RS}}{r_{NS}}\right)^3 = 3.3 \times 10^{21} \tag{12}$$

   which makes it possible to solve for the Roche limit radius without first solving for the densities of the two stars.] Applying the *expression for when tidal forces will overwhelm self-gravity, I can compute the Roche radius to be:*

$$\begin{aligned} d &= 2.44\left(\frac{\rho_M}{\rho_m}\right)^{1/3} R_M \\ &= 2.44\left(\frac{\rho_{NS}}{\rho_{RS}}\right)^{1/3} R_{NS} \\ &= 2.44\left(3.3\times10^{21}\right)^{1/3}(10^4 m) \\ &= 3.7\times10^{11} m \end{aligned}$$

Wow, $\boxed{\textit{the stars have to be separated by } 3.6\times10^{11}\ \textit{m or 2.4 AU}}$ *when the neutron star will start stripping matter off the Red Giant. Its pretty impressive the distance can be that huge!*

(b) **[10 points]** Let's assume mass transfer is ongoing, so the separation between the stars is what you derived in part (a). Let's also assume that their orbits are circular, so the semi-major axis of their orbits is equal to this separation. What would be the orbital period of these two stars?

Applying Kepler's Third Law, we find the orbital period:

$$\begin{aligned} (M_1+M_2)_\odot &= \frac{a_{AU}^3}{P_{years}^2} \\ P_{years} &= \sqrt{\frac{a_{AU}^3}{(M_1+M_2)_\odot}} \\ &= \sqrt{\frac{2.4^3}{4}} = 1.86 \end{aligned}$$

So the orbital period is $\boxed{\textit{1.86 years}}$.

(c) **[10 points]** Clearly explain why transfer of even a small fraction of the mass from the red giant to the neutron star would cause the neutron star to "spin up" (spin faster)? **Hint:** Consider where the angular momentum for the neutron star must come from.

It is the orbital angular momentum of the red giant, transfered when the mass from that star falls on the neutron star, causing its spin angular momentum to increase.

The mathematics for this are not too complex, but were unnecessary for full credit. However, I will lay them out here. Using the orbital period from part (b) and the separation between the stars from part (a), I can compute the orbital angular momentum of the red giant to be

$$\begin{aligned} L_{RG} &= I\omega \qquad (13)\\ &= MR^2\omega \\ &= \left(2\times 2\times 10^{30}kg\right)\left(3.6\times 10^{11}m\right)^2\left[\frac{2\pi}{1.86yr(3.15\times 10^7 s/yr)}\right] \\ &= 5.56\times 10^{46}\frac{kg\cdot m^2}{s} \qquad (14)\end{aligned}$$

So there is $5.56\times 10^{46}\frac{kg\cdot m^2}{s}$ *of orbital angular momentum, so we have a lot of spare orbital angular momentum to play with! Transferring even a little bit of it to the neutron star's spin should be enough to get it spinning considerably faster. Let's take* $1/10000$ *of this angular momentum and transfer it to a solid uniform sphere of 2 solar masses and 10 km radius. The moment of inertia of a solid sphere is* $I=\frac{2}{5}MR^2$*, therefore:*

$$\begin{aligned} L_{NS} &= I_{NS}\omega_{NS} = \frac{1}{10000}L_{RG} \\ \omega_{NS} &= \frac{L_{RG}}{10000 I_{NS}} = \frac{5L_{RG}}{20000 M_{NS}R_{NS}^2} \qquad (15)\\ &= \frac{5\left(5.56\times 10^{46}\frac{kg\cdot m^2}{s}\right)}{20000\left(4.00\times 10^{30}kg\right)\left(10000m\right)^2} \\ \omega_{NS} &= 34750s^{-1} \qquad (16)\end{aligned}$$

Fast enough to be a millisecond pulsar!!!

4. **[35 points total]** We discussed the ground state hyperfine transition of neutral hydrogen (HI) in one of our homework problems. In essence, due to the interaction of the magnetic moment of the electron and that of the proton in the nucleus, there is a slight energy difference when the two magnetic moments are parallel (lower energy) versus anti-parallel (higher energy), so the ground state of hydrogen splits into two hyperfine states. An electron in the higher energy state is metastable and decays with a half-life of approximately 10 million years. During World War II, Dutch astronomer Hendrick van der Hulst theorized the hyperfine transition should be visible and computed the frequency of the expected transition. It was first observed 7 years later in 1951.

   (a) **[10 points]** van der Hulst computed the expected frequency of the photons emitted during the HI hyperfine transition to be 1420.40575177 MHz based on the predicted energy difference between the two hyperfine states. Reverse what van der Hulst did and compute the energy difference between the two hyperfine states based on this predicted frequency.

   Given the frequency of the emitted photons, we can compute the energy very simply using Planck's expression for the energy of a photon:

$$\begin{aligned} E &= h\nu & (17) \\ &= \left(6.626 \times 10^{-34} J \cdot s\right)\left(1.42040575177 \times 10^{9} s^{-1}\right) & \\ E &= 9.412 \times 10^{-25} J & (18) \end{aligned}$$

   By conservation of energy, the energy per emitted photon must correspond to the energy difference between these two hyperfine states. So the energy difference between the two hyperfine states is $\boxed{9.412 \times 10^{-25} J \text{ or } 5.875 \times 10^{-6} eV!}$

   (b) **[13 points]** The minimum temperature in 'unshielded' parts of the universe is essentially the temperature of the Cosmic Microwave Background (CMB) radiation due to the Big Bang. Given the observed CMB temperature of 2.725 Kelvin, what fraction of the HI should be in the 'excited' hyperfine state? Clearly provide the reasoning for your answer. **HINT:** You can assume the statistical weights of these two levels are about equal to one another (e.g. - $g_2/g_1 \approx 1$).

Using Boltzmann's equation for the relative populations of two energy levels, and the information about g_2/g_1 *from the problem and the energy difference between the energy levels from part (a), I can compute that for a temperature of 2.725K, the relative populations of the two hyperfine energy levels should be*

$$\begin{aligned} \frac{n_2}{n_1} &= \frac{g_2}{g_1} \exp\left[-(E_2 - E_1)/kT\right] \qquad (19) \\ &= (1) \exp\left[-(9.412 \times 10^{-25} J)/(1.380 \times 10^{-23} \frac{J}{K})(2.725K)\right] \\ &= 0.975 \end{aligned}$$

So at any given time, about 97.5% of the hydrogen atoms should be in the excited hyperfine state, even if they are at just 2.725K! This is a testament to how small the energy difference between these two hyperfine states is, that a very cool temperature is enough to excite the state easily. I will point out real HI gas tends to be slightly warmer, about 10K. Given the half-life of this hyperfine state of over 10 million years, we might expect the hydrogen atoms to indeed get into thermal equilibrium with the CMB photons, so this should be an reasonable estimate, barring some huge error in the statistical weights.

(c) [**12 points**] Why did the discovery of the HI hyperfine spectral line lead van der Hulst and others to conclude most of the interstellar hydrogen must be very cool ($T < 100K$)? In a related vein of reasoning, briefly explain why we should not expect to see the hydrogen hyperfine transition in the regions around hot stars where lots of UV radiation is present.

Simply put, if the HI gas gets too warm, it no longer remains neutral hydrogen but instead becomes ionized hydrogen (a.k.a. "HII") via the process of collisional ionization. If there are too many UV photons in the environment (such as near a very hot star), instead of collision ionization, photoionization by UV photons can occur. In either case, you end up with HII, which has no hyperfine states since it has no electrons.